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References to reviews in Mathematical Reviews before volume 20 (1959) are by volume and page number, as MR **19**, 532: from volume 20 on, by volume and review number, as MR **20** #4387. Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

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Mathematical Reviews

Vol. 22, No. 4B April, 1961 Reviews 3021–3634

PROBABILITY

See also A2890, 3085, 3420, 3421, 3422, 3612.

3021:

Parzen, Emanuel. ★Modern probability theory and its applications. A Wiley Publication in Mathematical Statistics. John Wiley & Sons, Inc., New York-London, 1960. xv+464 pp. $10.75.

This book is one of the more substantial and ambitious additions in recent years to the rapidly growing text book literature in probability theory. It starts with simple combinatorial probability, emphasizing the construction of a finite sample space (Chapters 1 and 2) and follows the usual approach by going on to a discussion of independence and conditional probabilities, still for a finite sample space, including a brief discussion of finite Markov chains (Chapter 3). Chapters 4, 5, and 6 provide an elementary treatment of distribution functions, the mean and variance, and the normal, Poisson and related probability laws. These first six chapters, totalling 267 pages, are accompanied by hundreds of well chosen problems which rarely probe deeply but still acquaint the student with the amazing variety of possible applications. The treatment is extremely leisurely, perhaps oriented more toward the average than toward the very talented student. If it were not for their length one would readily agree with the author's estimate that "the first six chapters constitute a one quarter course in elementary probability theory at the sophomore or junior level. For the study of these chapters the student need have had only one year of college calculus. Students with more mathematical background would also cover chapters 7 and 8." Chapter 7 (random variables) and 8 (expectation of a random variable) constitute a careful treatment of real valued random variables as measurable functions on an abstract sample space, discussing the familiar difficulties arising due to a lack of an abstract theory of integration on the sample space. Thus, these chapters constitute essentially a repeat performance of topics introduced earlier in chapters 1 and 4. The psychological clash between the intuitive elementary treatment on the one hand, and the sophisticated yet still incomplete one on the other, is certainly an essential part of the student's learning process and growing pains, although it is unusual to find both combined within the covers of one book. However, this reviewer regrets the use of much new terminology and sometimes the use of more than one new term for variants of the same old idea. Thus in chapters 1 through 4 the probability measure is "a function of events on the subsets of a sample description space of a random phenomenon" (see p. 18 where this function is then required to satisfy the usual axioms). This terminology is used extensively up to chapter 7 when it becomes essentially useless ballast, and one hopes the student can discard the vocabulary while retaining the intuitive notions which it served to describe.

The last two chapters contain some of the author's own new proofs of classical limit theorems. "Chapters 9 and 10 are much less elementary in character than the first eight chapters. They constitute an introduction to the limit theorems of probability theory and to the role of characteristic functions in probability theory." Most of the proofs are new, in particular those of the inversion and continuity theorems and of an elegant form of the law of large numbers for dependent random variables.

F. L. Spitzer (Princeton, N.J.)

3022:

Miles, R. E. The complete amalgamation into blocks, by weighted means, of a finite set of real numbers. Biometrika **46** (1959), 317–327.

The problem considered is as follows. A set of n real numbers X_i, $i=1, \cdots, n$ (the ordinates) and a set of n positive real numbers W_i, $i=1, \cdots, n$ (the weights) are given. From these a set of $n!^2$ arrays is formed by permutations of the ordinate and weight indexes. If σ $(\sigma_1, \cdots, \sigma_n)$ is an ordinate permutation and $\sigma'(\sigma_1', \cdots, \sigma_n')$ a weight permutation, a typical array is

$$\begin{pmatrix} X_{\sigma_1}, \cdots, X_{\sigma_n} \\ W_{\sigma_1'}, \cdots, W_{\sigma_n'} \end{pmatrix}.$$

An amalgamation is an operation on an array by weighted averaging of consecutive entries; if the entries averaged are $X_{\sigma_i}, X_{\sigma_{i+1}}, \cdots, X_{\sigma_j}$, the amalgamation replaces them and the corresponding W's by X_j^i and W_j^i where

$$W_j^i X_j^i = W_{\sigma_i'} X_{\sigma_i} + W_{\sigma_{i+1}'} X_{\sigma_{i+1}} + \cdots + W_{\sigma_j'} X_{\sigma_j},$$
$$W_j^i = W_{\sigma_i'} + \cdots + W_{\sigma_j'}.$$

A complete amalgamation is a succession of such operations carried to the point where the first line entries are in strictly decreasing size from left to right. (To ensure strict decrease it is assumed that $X_r^1 \neq X_s^r$, $1 \leq r$, $s \leq n-1$, $2 \leq r+s \leq n$, for any permutations.) The entries in the completely amalgamated array are called blocks. How many arrays have a complete amalgamation into k blocks?

It is shown first that this answer is independent of both the ordinates and the weights. Next, if each array is chosen with equal probability $(n!^{-2})$, then $P(k, n)$, the probability of an array which is completely amalgamated in k blocks, is $|S_n^k|/n!$, with $|S_n^k|$ the signless Stirling number of the first kind. It is noted that this result is formally similar to that for enumeration of permutations by number of cycles and also to the enumeration of

permutations by "blocks". The latter is described as follows; the first block of a permutation of n consists of n and all its predecessors, the next block the largest remaining element and its predecessors, and so on. There is also a note added in proof recognizing the kinship of this problem with that considered by E. S. Andersen [Math. Scand. **2** (1954), 195–223; MR **16**, 839; pp. 209–218]. Finally an appendix gives the relation of the author's results to certain results on testing homogeneity of ordered alternatives given by D. J. Bartholomew [Biometrika **46** (1959), 36–48; MR **21** #3067].

J. Riordan (New York)

3023:

Gaede, Karl-Walter. Die Verteilung von gewissen reellen Wurzeln der Gleichung n-ten Grades mit reellen Zufallskoeffizienten. Monatsh. Math. **63** (1959), 359–367.

The author considers an algebraic equation

$$f(x) \equiv A_0x^n + A_1x^{n-1} + \cdots + A_n = 0$$

in which $A_k = U_j$, a random real variable, if $k = \nu_j$ ($j = 1, 2, \cdots, m$), where $0 \leq \nu_1 < \cdots < \nu_m \leq n$, $1 \leq m \leq n+1$, but otherwise A_k is a constant. He supposes that the distribution of $U_1, \cdots, U_m$ is determined by a joint probability density $g(u_1, \cdots, u_m)$ that is positive throughout a (closed) convex region L and vanishes outside L, and that the hyperplane of points $(u_1, \cdots, u_m)$ for which $f(x_1) = 0$ (with $U_j = u_j$) divides L into regions $K(x_1)$ and $\bar{K}(x_1)$ such that the integral $F(x_1)$ of $g(u_1, \cdots, u_m)$ over $K(x_1)$ is an increasing function of x_1. He proves that if there are two numbers α, β with $\alpha < \beta$, $F(\alpha) = 0$, $F(\beta) = 1$, such that the hyperplanes $f(\alpha) = 0$ and $f(\beta) = 0$ meet L, and if the hyperplanes $f(x_1) = 0$ and $f'(x_1) = 0$ have no common point in L when x_1 is in the interval $I: \alpha \leq x \leq \beta$, then the cumulative distribution function $\Phi_1(x_1)$ of the (sole) random real root X_1 in I of the equation $f(x) = 0$ is equal to $F(x_1)$ when x_1 is in I, to 0 when $x_1 < \alpha$, and to 1 when $x_1 > \beta$. He uses this theorem to determine $\Phi_1(x_1)$ in a numerical special case, in which $n = 5$, $m = 3$, $\nu_1 = 0$, $\nu_2 = 1$, $\nu_3 = 2$.

H. P. Mulholland (Exeter)

3024:

Sales Vallés, Francisco de A. On the continuity of the covariance of random functions of second order. Collect. Math. **11** (1959), 69–75. (Spanish)

The author defines the angular distance between two random variables of the second order (each having zero expectation) as $d(X, Y) = [1 - \mathrm{RE}(X, \bar{Y})\sigma_x^{-1}\sigma_y^{-1}]^{1/2}$, where $\bar{Y}$ is the complex conjugate of Y and $\mathrm{RE}(X, \bar{Y})$ is the real part of the expectation of $X \cdot \bar{Y}$. He then defines angular convergence of a sequence of random variables in the obvious way and proves theorems relating this type of convergence to convergence in the quadratic mean. For example, he proves that angular convergence, combined with convergence of the variances, implies convergence in quadratic mean. He applies these results to the study of the relationship between the continuity of a random function of the second order and the continuity of the covariance of that function and gives proofs of theorems on the continuity of the covariance due to Loève [*Probability theory*, Van Nostrand, New York, 1955; MR **16**, 598; pp. 468–470].

H. W. Brinkmann (Swarthmore, Pa.)

3025:

Bharucha-Reid, A. T. Über die Konvergenz der Folgen von verallgemeinerten zufälligen Grössen in Orlicz'schen Raümen. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 425–427. (Russian summary, unbound insert)

In this paper the author announces three theorems without proof about the weak and strong convergence of sequences of random variables in separable Orlicz spaces.

W. A. J. Luxemburg (Pasadena, Calif.)

3026:

Bartlett, M. S. ★An introduction to stochastic processes, with special reference to methods and applications. Cambridge University Press, New York, 1960. xiv+312 pp. $2.95.

Paperbound reprinting of first edition [1955; MR **16**, 939].

3027:

Бартлетт, М. С. [Bartlett, M. S.] ★Введение в теорию случайных процессов. [An introduction to the theory of stochastic processes]. Translated from the English by B. A. Sevast'yanov. Izdat. Inostr. Lit., Moscow, 1958. 384 pp. 15.25 r.

Translation of the 1955 edition [see #3026].

3028:

Johns, M. V., Jr.; Pyke, Ronald. On conditional expectation and quasi-rings. Pacific J. Math. **9** (1959), 715–722.

Es seien $(\Omega, \mathscr{A}, P)$ ein kompletter Wahrscheinlichkeitsraum, $\mathscr{E}$ der Raum aller P-integrierbaren Zufallsvariablen, $E \circ X = \int_\Omega X dP$, d.h. die Erwartung der Zufallsvariablen $X \in \mathscr{E}$, I_A die charakteristische Funktion (indicator function) von $A \in \mathscr{A}$. Bedeutet nun $\mathscr{F}$ einen kompletten σ-Unterkörper von $\mathscr{A}$, so wird bekanntlich die bedingte Erwartung $E\{X|\mathscr{F}\}$ von einer Zufallsvariablen $X \in \mathscr{E}$ bezüglich $\mathscr{F}$ definiert als die Klasse aller $\mathscr{F}$-messbaren Lösungen $Y \in \mathscr{E}$ des Systems der Gleichungen

$$(1) \qquad E \circ (X-Y)I_A = 0, \quad A \in \mathscr{F}.$$

Die Klasse $E\{X|\mathscr{F}\}$ der Lösungen ist nach dem Satz von Radon-Nikodym nicht leer. Verff. untersuchen nun, in wie weit man die Anzahl der Gleichungen (1) vermindern kann, ohne die Tragweite der Definition zu stören, d.h. ob man $\mathscr{F}$ durch ein Untersystem $\mathscr{S}$ von $\mathscr{F}$ ersetzen kann, derart dass aus der Gültigkeit von (1) für jedes $A \in \mathscr{S}$ die Gültigkeit von (1) für jedes $A \in \mathscr{F}$ auch folgt, und zeigen, dass dies der Fall ist, wenn $\mathscr{S}$ ein Quasi-Ring ist, der aber $\mathscr{F}$ erzeugt. Sie zeigen weiter dass aus der bedingten Unabhängigkeit von Quasi-Ringen die bedingte Unabhängigkeit der σ-Algebren (σ-Körpern), die sie erzeugen, folgt. Die Theorie der Verff. wird dann zur Bestimmung von äquivalenten Charakterisierungen des Begriffes der bedingten Unabhängigkeit angewandt, bzw. zum Beweise, dass gewisse stochastische Prozesse Markoff-Prozesse sind.

D. A. Kappos (Athens)

3029:

Belyaev, Yu. K. Analytic random processes. Teor. Veroyatnost. i Primenen. **4** (1959), 437–444. (Russian. English summary)

The author improves a result of the reviewer [Note in P. Lévy, *Processus stochastiques et mouvement Brownien*, Gauthier-Villars, 1948; MR **10**, 551]; he proves that if the covariance of a second order random function (r.f.) is analytic in a neighborhood of (t_0, t_0) then the same is true of almost all its sample functions. He then shows that the converse is also true for normal r.f.'s, and gives applications to second order stationary r.f.'s.

M. Loève (Berkeley, Calif.)

3030:

Maruyama, Gisirô; Tanaka, Hiroshi. Ergodic property of N-dimensional recurrent Markov processes. Mem. Fac. Sci. Kyushu Univ. Ser. A. **13** (1959), 157–172.

Consider a strict Markov process on R^n $(n \geqq 1)$ with right continuous sample paths $x(t)$ $(t \geqq 0)$; let $P_a(B)$ be the probability of the event B as a function of the starting point $a = x(0)$; let the motion be recurrent in the sense that $P[\sigma_D < +\infty] = 1$ if $D \subset R^n$ is open, σ_D being the entrance time $\inf(t : x(t) \in D)$; let

$$G_\alpha : f \to E\left[\int_0^{+\infty} e^{-\alpha t} f(x(t))dt\right], \quad \alpha > 0,$$

map $C(R^n)$ into itself; let $u(a) \equiv \int_D P_a[x(\sigma_D) \in db]f(b)$ be continuous outside D for each bounded Borel f; and let $P_a[\sigma_{D_1} < \sigma_{D_2}]$ be positive for each choice of open D_1, D_2 $(\bar{D}_1 \cap \bar{D}_2 = \emptyset)$ and each choice of a in a connected component D of $R^n - \bar{D}_1 \cup \bar{D}_2$ such that ∂D meets both ∂D_1 and ∂D_2.

Given D_1 and D_2 as above, introduce the successive entrance times

$$\sigma_1 = \sigma_{D_1}, \qquad \tau_1 = \inf(t : x(t) \in D_2, t > \sigma_1),$$
$$\sigma_2 = \inf(t : x(t) \in D_1, t > \tau_1), \quad \tau_2 = \inf(t : x(t) \in D_2, t > \sigma_2),$$

and so on; then the entrance places $x(\sigma_n)$ $(n \geqq 1)$ and $x(\tau_n)$ $(n \geqq 1)$ constitute ergodic Markov chains satisfying Doblin's condition, and, using the corresponding stable distributions Π_1 and Π_2, it turns out that

$$m(B) = \int_{D_1} \Pi_1(da) E_a [\text{meas}\,(t : x(t) \in B, t < \sigma_{D_2})]$$
$$+ \int_{D_2} \Pi_2(da) E_a [\text{meas}\,(t : x(t) \in B, t < \sigma_{D_1})]$$

is stable for the original motion, i.e.,

$$m(B) = \int_{R^n} m(da) P_a[x(t) \in B] \quad (t \geqq 0).$$

m is the only such stable mass distribution up to a multiplicative constant, and, as in the case of the 2-dimensional Brownian motion [see C. Derman, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 1155–1158; MR **16**, 495], a Hopf ergodic theorem holds.

T. Ueno has obtained these and further results and will publish them soon in Kōdai Math. Sem. Rep.; for additional information on stable mass distributions, see T. E. Harris [Proc. 3rd. Berkeley Sympos. Math. Statist. Probability, 1954–1955, vol. II, pp. 113–124; Univ. Calif. Press, Berkeley, Calif., 1956; MR **18**, 941] and E. Nelson [Duke Math. J. **25** (1958), 671–690; MR **21** #365].

H. P. McKean, Jr. (Cambridge, Mass.)

3031:

Yadrenko, M. Ĭ. Isotropic Gauss random fields of the Markov type on a sphere. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 231–236. (Ukrainian. Russian and English summaries)

Author's summary: "The author describes the random fields, indicated in the heading, on a sphere S_m in $(m+1)$-dimensional space and on a sphere S_∞ in Hilbert space possessing, in addition, some property of "Markovness". In the case of $m = 2$, this property is reduced to the requirement that whatever the curve K dividing S_2 into two parts may be, and whatsoever the points P_1 and P_2 separated by K, the random functions $\xi(P_1)$ and $\xi(P_2)$ are independent if the values $\xi(P)$ on K are known."

3032:

Bharucha-Reid, A. T. ★Elements of the theory of Markov processes and their applications. McGraw-Hill Series in Probability and Statistics. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1960. xi + 468 pp. $11.50.

As the following chapter headings indicate, considerably over half of the book is devoted to applications. (1) Processes discrete in space and time. (2) Processes discrete in space and continuous in time. (3) Processes continuous in space and time. (The remaining chapters are 'Applications in —'.) (4) Biology. (5) Physics: Theory of cascade processes. (6) Physics: Additional applications. (7) Astronomy and astrophysics. (8) Chemistry. (9) Operations research: The theory of queues.

There are appendices on: (A) generating functions; (B) the Laplace and Mellin transforms; (C) Monte Carlo methods in the study of stochastic processes. The theoretical chapters contain problems, and all chapters are followed by voluminous bibliographies.

The book is written as a compendium rather than as a complete treatment of the topics covered. The stress is on manipulation rather than ideas. Unfortunately the theoretical section contains many mathematical and typographical slips. For example (p. 14) according to the definition given, all states of a countable state space Markov chain are periodic. The analysis of the asymptotic character of the probability $p_{ij}^{(n)}$ of a transition from state i to state j in n steps is easily reduced, as usual, to the key case $i = j$, which case is then treated simply by 'Now $p_{jj}^{(n)} \to \pi_j$' with no indication that the quoted assertion is not immediately obvious. (In fact it is the only difficult part of the analysis.) The definitions of discrete branching processes and of Markov chains (pp. 10, 11) are intermingled in a confusing way, and the former definition is incorrect.

The treatment of the applications is more satisfactory, since, at least as treated, the essentials are the formal manipulations and techniques needed to derive specific results. Many mathematicians and applied workers will find the large collection of applications useful.

J. L. Doob (Urbana, Ill.)

3033:

Heathcote, C. R.; Moyal, J. E. The random walk [in continuous time] and its application to the theory of queues. Biometrika **46** (1959), 400–411.

The authors consider the random walk in continuous time with the 'backward' Kolmogorov equations,

$$P'_{k,n}(t) = -(\lambda+\mu)P_{k,n}(t)+\lambda P_{k+1,n}(t)+\mu P_{k-1,n}(t),$$

λ and μ being positive constants. The generating function $G_k(z, t)=\sum_n P_{k,n}(t)z^n$ satisfies a very similar equation, and its Laplace transform $g_k(z, s)$ satisfies

$$(s+\lambda+\mu)g_k = \lambda g_{k+1}+\mu g_{k-1}+z^k.$$

The authors solve this difference equation for a variety of boundary conditions, and so (after a Laplace inversion) obtain explicit formulae for the transition probabilities in the following cases: (1) No boundary conditions (the unrestricted random walk); (2) two absorbing barriers; (3) one absorbing barrier; (4) two reflecting barriers (i.e., the queue $M/M/1$ with a waiting room of finite size; (5) one reflecting barrier (i.e., $M/M/1$ with no such restriction). By a limiting process, analogous diffusion solutions are found in each case. Finally the authors modify the problem by (a) allowing λ and μ to depend linearly on k, and also (b) by keeping λ and μ constant save that μ is to be proportional to k for $0\leq k\leq N$ (this is the queue $M/M/s$, with $s=N$). The method of solution in case (a) is sketched, and then the special case (b) is worked out in full with (6) one reflecting barrier and (7) one absorbing barrier, thus obtaining formulae relating to the transient behaviour of $M/M/s$, and its behaviour within a "busy period". As the authors point out, many of these formulae have been obtained previously, but some appear to be new, and this systematic and unified account will be found most useful.

D. G. Kendall (Oxford)

3034:

Koenigsberg, Ernest. Finite queues and cyclic queues. Operations Res. 8 (1960), 246–253.

The author considers the problem of a system with N machines (to perform some sort of service), A ($A\leq N$) machine operators, and M repair stations. The problem is to determine the various average utilities of the machines assuming both an exponential individual machine breakdown rate, μ_2, and an exponential individual machine repair rate, μ_1. Using the standard procedures, the author sets up the steady state probability equations and obtains solutions for the mean number of units either available for use or in repair, the mean number actually in use, and the mean lengths of waiting lines before the work stations and before the repair stations. He presents tables for different (small) values of M, N, and A and for $\mu_2/\mu_1=0.25$ showing the effect on the different mean values calculated and on derived measures of utility. Finally, he shows in another table how additions to the repair or operating facilities, to the number of machines, or to the repair rate affects the utility of the system.

H. M. Gurk (Princeton, N.J.)

3035:

Prabhu, N. U. Some results for the queue with Poisson arrivals. J. Roy. Statist. Soc. Ser. B **22** (1960), 104–107.

Customers arrive at a counter according to a Poisson process of density λ. There is a single server. The service times have a general distribution function $B(x)$. Let z be the initial occupation time of the server at time $t=0$. Denote by T the time when the server will be idle for the first time after $t=0$ and by N the number of customers arriving at the counter during the time interval $(0, T)$. The author proves that

$$P(T \leq t, N = n) = \lambda z \int_z^t e^{-\lambda u}\frac{(\lambda u)^{n-1}}{n!}\,dB_n(u-z) \text{ if } t \geq z,$$
$$= 0 \quad \text{if } t<z,$$

where $B_n(x)$ is the n-fold convolution of $B(x)$ with itself and $B_0(x)=1$ if $x\geq 0$, $B_0(x)=0$ if $x<0$. Using this formula the author determines the distribution function of the length of the busy period and the distribution of the number of customers served during a busy period.

L. Takács (New York)

3036:

Beneš, V. E. General stochastic processes in traffic systems with one server. Bell System Tech. J. **39** (1960), 127–160.

This extremely interesting paper is bristling with original ideas. The author considers the general one-server traffic systems in which (I) a rejected customer waits for service or (II) he is permanently lost to the system. In case I he writes $K(t)$ for the cumulative load offered up to epoch t, and he writes $W(t)$ for the virtual waiting time at epoch t, with $W(0)=K(0)$. He introduces the kernel

$$(1) \qquad R(t, u, w) = \text{pr}\,\{K(t)-K(u)-t+u \leq w \mid W(u) = 0\},$$

and proves that

$$(2) \quad \text{pr}\,\{W(t) \leq w\} = \text{pr}\,\{K(t)-t \leq w\} - \frac{\partial}{\partial w}\int_0^t R(t, u, w)\,\text{pr}\,\{W(u) = 0\}du,$$

while $\text{pr}\,\{W(u)=0\}$ can be found as solution to the integral equation

$$\int_0^{t+w} \text{pr}\,\{K(t) \leq u\}du = \int_0^{t+w} R(t, u, w)\,\text{pr}\,\{W(u) = 0\}du.$$

It is to be emphasised that absolutely no assumptions are made concerning the stochastic structure of the random function $K(t)$ other than the requirement that it is to be a nondecreasing step-function, and that the various probabilities, etc., are to be well-defined. As an example of the use of (2) and the following integral equation, the author then considers the exceedingly special case of the $M/G/1$ queue, and here he shows by a complicated analytical argument that

$$\text{pr}\,\{W(t) = 0 \mid W(0)\} = t^{-1}\mathscr{E}\{[t-K(t)]^+\} + t^{-1}W(0)\,\text{pr}\,\{K(t) \leq t\}.$$

A direct probabilistic derivation of this would be worth seeking.

The author then turns to loss systems (case II). $K(t)$ is defined as before, and now $A(t)$ is the unexpended service time of the customer (if there is one) being served at the epoch t; if the system is free, then $A=0$. Kernels R and Q analogous to (1) are defined by formulae which can be loosely paraphrased as follows: $R(t, u)$ is pr {service time of a successful customer arriving at u is $\leq t-u$, given that a s.c. did arrive at u}; $Q(t, u)=$ pr {next successful arrival

after u occurs before t, given that a s.c. did arrive at u}. He then shows that

$$\operatorname{pr}\{A(t) \leq w\} = \operatorname{pr}\{A(0) \leq t+w\} - \int_0^t [1-R(t+w, u)]d\mathscr{E}S(u),$$

where $\mathscr{E}S(u)$ satisfies the integral equation,

$$\mathscr{E}S(t) = \operatorname{pr}\{y_1 \leq t\} + \int_0^t Q(t, u)d\mathscr{E}S(u);$$

here y_1 is the arrival epoch of the first successful customer, and $d\mathscr{E}S(u)$ is the expected number of arrivals of s.c.'s in $(u, u+du)$. Finally the author shows that when the kernels $R(t, u)$ and $Q(t, u)$ depend only on the difference $(t-u)$, and when one or two other conditions hold, then $\lim_{t\to\infty} \operatorname{pr}\{A(t) \leq w\}$ exists, and he evaluates this limit.

D. G. Kendall (Oxford)

3037:

Beneš, V. E. Combinatory methods and stochastic Kolmogorov equations in the theory of queues with one server. Trans. Amer. Math. Soc. **94** (1960), 282–294.

This paper presents a new approach to the study of delays in a single-server queueing system, that is, one in which the order service is "first come, first served" (strict queueing) and with no defections from the queue. The delays are represented by a single variable, the virtual waiting time, $W(t)$ (the waiting time of actual demands at arrival epochs and of fictional demands otherwise). The starting point is a functional equation relating $W(t)$ to a function $K(t)$ which represents the (cumulative) work-load of the system; $K(t)$ is a staircase function with risers at arrival epochs, of sizes equal to the corresponding service times. There are no assumptions of independence (of service times or interarrival times) or of special distributions for these random variables. It is shown first that $W(t)$ is a well-defined stochastic process. Next $\exp[-sW(t)]$ is given a relatively simple expression, by means of a combinatory argument, in terms of $K(u)-u$, $u \leq t$, and the (characteristic) function $P(u, 0)$ of $W(u)$ ($P(u, 0)=1$ if $W(0)=0$, $P(u, 0)=0$ if $W(0)\neq 0$; namely

$$\exp[-sW(t)] = \exp[-s(K(t)-t)][1-s\int_0^t \exp[s(K(u)-u)]P(u, 0)du].$$

This is the key to the proof of a succession of theorems which culminates in the determination of "the form of an operator which gives the distribution of $W(t)$ in terms of distributions associated with $K(u)$ for $u \leq t$. The operator is linear, and acts only on the distribution of $K(t)-t$, and, for each $u \leq t$, on the conditional distribution of $K(t)-K(u)-t+u$ relative to the knowledge that $\sup_{0\leq y\leq u}[K(u)-K(y)-u+y] \leq 0$".

J. Riordan (New York)

3038:

Wilkins, Coleridge A. On two queues in parallel. Biometrika **47** (1960), 198–199.

The author says, "In a paper published in this journal, Haight [Biometrika **45** (1958), 401–410; MR **20** #6737] investigated a two-queue system in which an arrival is assigned to the shorter queue, or if they are of equal length, to a particular one, the 'near' queue. The purpose of this note is to show that Dr. Haight's results can be extended to the more general case where, if X is the length of the near queue, Y that of the other, and $W(X, Y)$ is the probability of an arrival's joining the near one, we have $W(x, y)=1$ (if $x<y$), $w(x)$ (if $x=y$), 0 (if $x>y$)."

D. G. Kendall (Oxford)

3039:

Yüh, M. I. On the problem $M/M/n$ in the theory of queues. Acta Math. Sinica **9** (1959), 494–502. (Chinese. English summary)

From the summary: As usual, we use $M/M/n$ to denote a queueing process: the arrival and service time are assumed negative exponentially distributed with means λ^{-1} and μ^{-1} respectively, the number of servers is n. Let $p_k(t)$ be the chance that at time t there are k customers present including those being served. Theorem 1: For $k \geq n$, we have

$$p_k(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{ut}\alpha^k}{u\varphi(u)}\,du \quad (c>0),$$

where α is the root of $n\mu x^2-(\lambda+u+n\mu)x+\lambda=0$ with smaller absolute value, and

$$\varphi(u) = (1-\alpha)^{-1} - \mu\lambda^{-1}\alpha\sum_{j=0}^{n-2}(n-1-j)\alpha^j \times \sum_{i=0}^{j}\binom{j}{i}\lambda^{-i}(u+\mu)\cdots(u+i\mu).$$

Theorem 2: For $n=1$, 2 and 3, the limits $\lim_{t\to\infty} p_k(t)$ $(k=0, 1, 2, \cdots)$ exist.

3040:

Yüh, M. I. On the problem $M/M/s$ in the theory of queues. Sci. Record (N.S.) **3** (1959), 614–615.

Abstract of #3039.

3041:

Burrows, C. Some numerical results for waiting times in the queue $E_k/M/1$. Biometrika **47** (1960), 202–203.

In the reviewer's analysis [Ann. Math. Statist. **24** (1953), 338–354; MR **15**, 44] of the queueing system $GI/M/s$, the principal quantities of interest were evaluated in terms of a parameter λ. The author calculates λ for $E_k/M/1$, and shows that it is then given by (*) $[\rho k/(\rho k+1-\lambda)]^k=\lambda$ (this formula is misprinted in the paper). The author solves the equation (*) for $\rho=0.05$, 0.1 (0.1) 0.9 and $k=1$, 1.5, 2, 3, 4, 5, 6, 11, 16, 21, ∞. (Here ρ as usual denotes the traffic intensity.) He is then able to tabulate, for the same value of ρ and k, the figure of demerit $R=\mathscr{E}(w)/\mathscr{E}(v)$ in the reviewer's notation; in the special case considered here $R=\lambda/(1-\lambda)$. He points out that a change from $M/M/1$ via $E_k/M/1$ towards $D/M/1$ can bring about a larger saving in waiting time than can a change from $M/M/1$ via $M/E_k/1$ towards $M/D/1$; the contrast is particularly marked when the traffic intensity is small. When the traffic intensity is close to unity, on the other hand, these two modifications of $M/M/1$ have

nearly the same effect on the figure of demerit. In this connexion the reviewer would like to point out that

$$\lim_{\rho\uparrow 1}(1-\rho)R = \tfrac{1}{2}\{1+\operatorname{var}(v/b)\} \quad \text{for} \quad M/G/1$$

and that

$$\lim_{\rho\uparrow 1}(1-\rho)R = \tfrac{1}{2}\{1+\operatorname{var}(u/a)\} \quad \text{for} \quad GI/M/1;$$

here $u(a)$ is the (mean) inter-arrival time and $v(b)$ is the (mean) service time. *D. G. Kendall* (Oxford)

3042:

Finch, P. D. A probability limit theorem with application to a generalization of queueing theory. Acta Math. Acad. Sci. Hungar. **10** (1959), 317–325. (Russian summary, unbound insert)

Author's summary: "The fundamental theorem of this paper is the following. Theorem 1: Let $\{u_r\}$ be a sequence of independently and identically distributed random variables with common distribution function $U(x)$ such that $M(|u|)=\int_{-\infty}^{+\infty}|x|dU(x)<\infty$. Let v_r be a sequence of independently and identically distributed non-negative random variables with common distribution function $V(x)$ and such that $M(v)=\int_0^\infty x dV(x)<\infty$. Let the sequence $\{v_r\}$ be independent of the sequence $\{u_r\}$. Define a sequence of non-negative random variables $\{w_r\}$ $(r\geq 1)$ by the equations

$$(1)\qquad w_{r+1} \begin{aligned} &= w_r+u_r \quad \text{if} \quad w_r+u_r>0,\\ &= v_{r+1} \quad \text{if} \quad w_r+u_r \leq 0,\end{aligned}$$

for $r=0,1,2,\cdots$, where w_0 is a given non-negative random variable with d.f. $W_0(x)$. Write $W_r(x)=P(w_r\leq x)$ $(r=1,2,\cdots)$; then $W(x)=\lim_{r\to\infty}W_r(x)$ exists. If $M(u)\geq 0$, then $W(x)\equiv 0$. If $M(u)<0$, then $W(x)$ is the distribution function of a non-negative random variable, that is, $W(x)$ is a non-decreasing function of x and $\lim_{x\to\infty}W(x)=1$; further $W(x)$ is independent of $W_0(x)$ and is the unique (d.f.) solution to the following integral equation:

$$(2)\qquad \begin{aligned} W(x) &= 0 \qquad\qquad \text{if} \quad x<0,\\ &= \int_{-\infty}^{x} W(x-y)dU(y)\\ &\quad -\{1-V(x)\}\int_{-\infty}^{0} W(-y)dU(y) \quad \text{if} \quad x\geq 0.\text{"}\end{aligned}$$

J. Wolfowitz (Ithaca, N.Y.)

3043:

Finch, P. D. On the distribution of queue size in queueing problems. Acta Math. Acad. Sci. Hungar. **10** (1959), 327–336. (Russian summary, unbound insert)

The author treats a multi-server queue with general input and output distributions, and obtains a number of results about steady state (limiting) distributions which are not readily summarizable here and which complement those of Kiefer and the reviewer [Trans. Amer. Math. Soc. **78** (1955), 1–18; MR **16**, 601].

J. Wolfowitz (Ithaca, N.Y.)

3044:

Wang, Tzu-kwen. On a birth and death process. Sci. Record (N.S.) **3** (1959), 331–334.

It is shown that for a birth and death process the return from infinity can be regarded as the limiting case of return from a state with a large index, this state being instantly reached at the "first transcendent jump".

K. L. Chung (Syracuse, N.Y.)

3045:

Chow, Y. S. A martingale inequality and the law of large numbers. Proc. Amer. Math. Soc. **11** (1960), 107–111.

Let $\{y_k, k\geq 1\}$ be a semimartingale, let $c_1\geq c_2\geq\cdots$ be positive constants, and let $\varepsilon>0$. Then it is proved that

$$\varepsilon P\{\max_{m\geq k\geq 1} c_k y_k \geq \varepsilon\} \leq \sum_{1}^{m-1}(c_k-c_{k+1})E\{y_k^+\}+c_m E\{y_m^+\}.$$

Here $y^+=\max[y,0]$. If $c_k=1$ for all k, the inequality reduces to one proved by the reviewer [*Stochastic processes*, Wiley, New York, 1953; MR **15**, 445]. If y_k is the square of the sum of k random variables with zero means and finite variances, the inequality reduces to one proved by Hájek and Rényi [Acta Math. Acad. Sci. Hungar. **6** (1955), 281–283; MR **17**, 864], a generalization of a classical inequality of Kolmogorov. The author applies his inequality to derive versions of the strong law of large numbers proved by Brunk, Chung, Kawata and Udagawa, Kolmogorov, Prohorov for independent summands, and by Lévy and Loève for dependent summands.

J. L. Doob (Urbana, Ill.)

3046:

Levinson, Norman. Limiting theorems for age-dependent branching processes. Illinois J. Math. **4** (1960), 100–118.

The author considers the age-dependent branching process introduced by R. Bellman and the reviewer [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 601–604; MR **10**, 311]. The generating function for the number of particles at time t satisfies

$$F(s,t) = \int_{0-}^{t} h[F(s,t-y)]dG(y)+s[1-G(t)],$$

where $h(s)=\sum_{j=0}^\infty q_j s^j$, q_j is the probability that at the end of its life a particle is transformed into r particles, and $G(t)$ is the cumulative distribution function of the life length of an object. It is assumed that $1<\mu=h'(1)<\infty$, and that G has a density $G'(t)=g(t)$. If $Z(t)$ is the number of particles existing at t, then $m(t)=E[Z(t)]=\partial F(1,t)/\partial s$ satisfies an equation of the renewal type, whence is deduced the asymptotic behavior $\lim e^{-at}m(t)=c>0$, where a is defined by $\mu\int_0^\infty e^{-at}g(t)dt=1$. The principle result is the following. Let $\beta(t)$ be a continuous monotone decreasing function such that $\int_1^\infty[\beta(t)/t]dt<\infty$ and suppose $\sum_{j=n}^\infty jq_j\leq\beta(n)$. This is weaker than requiring a second moment, which would correspond to $\beta(n)=c/n$. Then $\lim_{t\to\infty}F(e^{-s/m(t)},t)=\phi(s)$ exists for $\operatorname{Re} s\geq 0$, $\phi(s)$ being the Laplace transform of the limiting probability law of $Z(t)/m(t)$. The function ϕ satisfies $\phi(s)=\int_0^\infty h[\phi(se^{-ay})]dG(y)$, $\phi(0)=1$, $\phi'(0)=-1$ and ϕ is the only continuous function bounded by 1 that satisfies these conditions. This generalizes a similar result for the Galton-Watson process proved by the author [same J. **3** (1959), 554–565; MR **21** #6637]. A similar result for the case $h(s^2)$ was proved by R. Bellman and the reviewer [loc. cit. and Ann. of Math. (2) **55** (1952), 280–295; MR **13**, 664].

T. E. Harris (Santa Monica, Calif.)

3047:

Feller, William. The birth and death processes as diffusion processes. J. Math. Pures Appl. (9) **38** (1959), 301–345.

The Kolmogorov differential equations (I) $P'(t)=AP(t)$, $P'(t)=P(t)A$, for the transition probabilities $P(t)=\{p_{ij}(t): i, j=0, 1, 2, \cdots\}$ for the birth and death process with birth and death rates β_i, δ_i dependent on the population size i in any manner, so that

$$A=\begin{bmatrix} -(\delta_0+\beta_0) & \beta_0 & 0 & 0 & \cdots \\ \delta_1 & -(\delta_1+\beta_1) & \beta_1 & 0 & \cdots \\ 0 & \delta_2 & -(\delta_2+\beta_2) & \beta_2 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

with $\delta_0\geq 0$ and all other β's and δ's positive, have been studied by a number of writers with widely differing points of view [W. Ledermann and G. E. H. Reuter, Philos. Trans. Roy. Soc. London Ser. A **246** (1954), 321–369; MR **15**, 625; S. Karlin and J. L. McGregor, Trans. Amer. Math. Soc. **85** (1957), 489–546; MR **19**, 989; G. E. H. Reuter, Acta. Math. **97** (1957), 1–46; MR **21** #918; and B. O. Koopman (to appear)]. In this memoir the author returns to the subject which he originated [Acta. Bioth. Ser. A. **5** (1939), 11–40; MR **1**, 22] in 1939, and gives an extensive and almost complete account of the whole problem of constructing solutions to (I) from a characteristic point of view which involves the re-interpretation of the matrix A as a sort of discrete Sturm-Liouville operator. The methods of the author's long series of papers on diffusion theory are followed closely, but the presentation is self-contained, and in fact will be welcomed as an illuminating introduction to the diffusion theory itself. The author's first trick is to define $x_0=1/\delta_0$ $(\delta_0>0)$ $[x_0=0$ if $\delta_0=0]$, $x_1=x_0+1/\beta_0$, $x_2=x_1+\delta_1/(\beta_0\beta_1)$, $\cdots$, $x_\infty=\lim x_n$, and to treat the set of points $E=\{x_0, x_1, x_2, \cdots\}$ and in some cases $\bar{E}=E\cup(x_\infty)$ as the natural state-space for the process. He writes $f^+(x_n)=[f(x_{n+1})-f(x_n)]/[x_{n+1}-x_n]$, with a suitable modification at $n=-1$. Next he imposes a measure on E by attaching a weight $\mu_n=(\beta_0\beta_1\cdots\beta_{n-1})/(\delta_1\delta_2\cdots\delta_n)$ (with $\mu_0=1$) to x_n, and he puts $D_\mu g(x_n)=[g(x_n)-g(x_{n-1})]/\mu_n$. If $p_{ij}(t)$ is one of the transition probabilities, he puts $p(x_i, x_j; t)=p_{ij}(t)/\mu_j$. It then turns out that if we have a family of Markovian matrices whose rows (or columns) satisfy the forward (or backward) equations then the rows (columns) of $\{p(x_i, x_j; t)\}$ satisfy (II) $(d/dt)f=D_\mu f^+(=\Omega f)$. This last equation is now treated as a diffusion equation, following the author's method [Illinois J. Math. **1** (1957), 459–504; MR **19**, 1052] for the diffusion theory. The boundary point x_∞ is described as (1) regular when $x_\infty<\infty$ and $\sum\mu_k<\infty$, (2) exit when not regular and $x_\infty<\infty$, $\sum(x_\infty-x_k)\mu_k<\infty$, (3) entrance when not regular but $\sum x_k\mu_k<\infty$, and (4) natural otherwise. Discrimination is also provided by the behaviour of the unique solution u to the equation $D_\mu u^+=\lambda u$ $(u(x_0)=1,\ \lambda>0)$; $u(x_\infty)<\infty$ if and only if x_∞ is regular or exit, while $\sum u(x_k)\mu_k<\infty$ if and only if x_∞ is regular or entrance. This function u is strictly increasing; the equation $D_\mu u^+=\lambda u$ $(x_k\neq x_0)$ also has a decreasing solution v satisfying $vu^+-uv^+=1$, and a Green's function is defined by $\pi_\lambda(x_i, x_k)=u(x_i)v(x_k)$ (if $i\leq k$), $=v(x_i)u(x_k)$ (if $i\geq k$). It is shown that $\pi_\lambda=\int_0^\infty e^{-\lambda t}p(x_i, x_k; t)dt$ for a "minimal" Markovian family of matrices satisfying both the forward and backward equations. In case (4) neither equation admits any other solution, and we have honesty if and only if $\delta_0=0$. In case (2) the forward equation has no other solution, but there are infinitely many solutions to the backward equation. In case (3) we have the converse situation. In case (1) (regular boundary) "complete non-uniqueness prevails".

The author then proceeds to construct Markovian solutions to the differential equations both for the original state-space E and for the augmented state-space $\bar{E}$. He remarks that his constructions are exhaustive in some cases, but not in all. The solutions associated with $\bar{E}$ provide interesting examples of systems with an instantaneous state, etc., and supplement other known examples of "pathological" (i.e., typical) behaviour.

Finally some probabilistic interpretations are given for the mathematical objects (u, v, the x's, etc.) occurring in the investigation, and this leads to probabilistic descriptions of the different sorts of boundary point and of the different solutions. *D. G. Kendall* (Oxford)

3048:

Gans, Paul J. Open first-order stochastic processes. J. Chem. Phys. **33** (1960), 691–694.

Author's summary: "The solution of the general open first-order stochastic process representing the relaxation of an open multistate system is obtained by the generalization of a method of Krieger and Gans. The result is valid for a system containing an arbitrary number of absorbing and semiabsorbing states as well as allowing the presence of any number of emitting states. The stability of the solution is discussed in terms of the matrix of the transition probabilities. This is done with the aid of a theorem of Lévy and Hadamard. The result is applied to the corresponding closed system as well as the present work."

STATISTICS

See also 3022, 3026, 3027, 3420, 3421, 3422, 3588.

3049:

Sarkadi, K. A rule of dualism in mathematical statistics. Acta Math. Acad. Sci. Hungar. **11** (1960), 83–92. (Russian summary, unbound insert)

This is essentially an English version of the author's earlier papers [Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 275–286, 287–297; MR **16**, 384]. The content of the second paper is here somewhat condensed.

E. Lukacs (Washington, D.C.)

3050:

Kellerer, Hans G. Lageparameter *n*-dimensionaler Verteilungsfunktionen. Math. Z. **73** (1960), 197–218.

A location parameter of an n-dimensional probability distribution is defined by a system of axioms. Special location parameters such as the first moment vector and an n-dimensional median are characterized by additional conditions. *W. Hoeffding* (Chapel Hill, N.C.)

3051:

Hogg, Robert V. Certain uncorrelated statistics. J. Amer. Statist. Assoc. **55** (1960), 265–267.

Define odd and even location functions $T(x)$, $S(x)$ by $T(x+he)=T(x)+h$, $T(-x)=-T(x)$, $S(x+he)=S(x)$, $S(-x)=S(x)$, for all real h, $x=(x_1, \cdots, x_n)$, where $e=(1, \cdots, 1)$. The authors prove that the symmetry of a p.d.f. implies that the correlation between an even and odd location statistic is zero, which generalizes a result of Ostle and Steck [same J. **54** (1959), 465–471; MR **21** #4491]. *I. Olkin* (Minneapolis, Minn.)

3052:

Blackwell, David; Hodges, J. L., Jr. The probability in the extreme tail of a convolution. Ann. Math. Statist. **30** (1959), 1113–1120.

Let $x, x_1, x_2, \cdots$ be independent integer-valued random variables having identical distributions, and let $\varphi(a, t)=\mathscr{E}(e^{t(x-a)})$, where a is any number on the interval $(\mathscr{E}(x), \sup x)$. Let $m(a)=\varphi(a, t(a))$, where $t(a)$ is the value of t which minimizes $\varphi(a, t)$ with respect to t. (There is a unique value of $t(a)$.) Let $y, y_1, y_2, \cdots$ be independent random variables having identical distributions such that x and y have the same range of values with

$$P(y=x') = \frac{1}{m(a)} e^{t(a)\cdot(x'-a)}\cdot P(x=x').$$

Let μ_2, μ_3, μ_4 be the central moments of y. The authors show that

$$P(\sum_1^n x_i = na) = [m(a)]^n \cdot P(\sum_1^n y_i = na)$$

and from this that $P(\sum_1^n x_i=na)=P_n[1+O(1/n^2)]$, where

$$P_n = \frac{[m(a)]^n}{(2\pi n\mu_2)^{1/2}}\left[1+\frac{1}{8n}\left(\frac{\mu_4}{\mu_2^2}-3-\frac{5}{3}\frac{\mu_3^2}{\mu_2^3}\right)\right].$$

They also show that

$$P(\sum_1^n x_i \geq na) =$$

$$\frac{P_n}{1-z}\left\{1-\frac{1}{2n}\left[\frac{z(1-z)(\mu_3/\mu_2)+z(1+z)}{(1-z)^2\mu_2}\right]\right\}[1+O(1/n^2)],$$

where $z=e^{-t(a)}$. *S. S. Wilks* (Princeton, N.J.)

3053:

Aitchison, J.; Silvey, S. D. Maximum-likelihood estimation procedures and associated tests of significance. J. Roy. Statist. Soc. Ser. B **22** (1960), 154–171.

This is an expository paper which describes and compares in "practical" terms "unrestricted" maximum-likelihood estimation procedures and "restricted" maximum-likelihood ones, and their associated tests of significance. Some of the underlying theory was developed in papers by Aitchison and Silvey [Ann. Math. Statist. **29** (1958), 813–828; MR **20** #1382], Silvey [ibid. **30** (1959), 389–407; MR **20** #3062], and Wald [Trans. Amer. Math. Soc. **54** (1943), 426–482; MR **7**, 20]. *M. Dwass* (Evanston, Ill.)

3054:

Masuyama, Motosaburo. Table of n, $\log_e n$, $n\log_e n$ and $n(\log_e n)^2$ for $n=1$ through 500 with applications. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **7** (1960), 56–64.

The author gives the formula for the entropy H of a multinomial distribution, the likelihood estimate of the entropy $\hat{H}$, the variance of the estimate $\sigma_{\hat{H}}^2$ and the large sample value of the likelihood estimate of the variance $\hat{\sigma}_{\hat{H}}^2$. The examples illustrate the computation of $\hat{H}$, $\hat{\sigma}_{\hat{H}}^2$, as well as their variations in time. There is also an example illustrating the application of the entropy (information) results to test a null hypothesis of independence in a two-way contingency table, and an example illustrating the test of a null hypothesis specifying a relation among the probabilities. The tables are for n: 1(1)500, $\log_e n$ to 5D, $n\log_e n$ to 1D, $n(\log_e n)^2$ to 1D. *S. Kullback* (Washington, D.C.)

3055:

Bašarin, G. P. On a statistical estimate for the entropy of a sequence of independent random variables. Teor. Veroyatnost. i Primenen. **4** (1959), 361–364. (Russian. English summary)

Let $\hat{p}_i$ be the relative frequency with which the letter i occurs in N mutually independent, identically distributed samples from a finite alphabet. The estimate $\hat{H}(N)=-\sum \hat{p}_i \log_2 \hat{p}_i$ for the entropy is shown to be biased, consistent, and asymptotically normal. *V. E. Beneš* (Murray Hill, N.J.)

3056:

Neyman, Jerzy. Optimal asymptotic tests of composite statistical hypotheses. Probability and statistics: The Harald Cramér volume (edited by Ulf Grenander), pp. 213–234. Almqvist & Wiksell, Stockholm; John Wiley & Sons, New York; 1959. 434 pp. $12.50.

A contribution to the problem of constructing widely applicable tests of composite parametric hypotheses based on a large number, n, of identically distributed variates. Suppose ϑ is a vector of s nuisance parameters ϑ_i which possess consistent estimates of a specified type when $\xi=\xi_0$. The author introduces a class $C(\alpha)$ of asymptotic tests of hypotheses about a parameter ξ of a probability density or frequency function $p(x|\xi, \vartheta)$ whose logarithmic derivatives, $\varphi_\xi(x, \vartheta)$ and $\varphi_{\vartheta_j}(x, \vartheta)$, with respect to ξ and ϑ_j respectively, satisfy certain regularity conditions. He proves, inter alia, that the test of the hypothesis $\xi=0$ against alternatives $\xi>0$ which possesses optimum power at (ξ_n, ϑ), where the sequence ξ_n converges to zero from the right ($\xi_n\sqrt{n}$ remaining bounded), is provided by the criterion

$$n^{-1/2}\sigma_0^{-1}(\hat{\vartheta}) \sum_{i=1}^n \{\varphi_\xi(x_i, \hat{\vartheta}) - \sum_{j=1}^s a_j^0 \varphi_{\vartheta_j}(x_i, \hat{\vartheta})\} > \nu(\alpha),$$

where $\{a_j^0\}$ is the set of regression coefficients that minimizes the variance of the random variable in braces, said minimum being $\sigma_0^2(\hat{\vartheta})$ when $\xi=0$, and $\hat{\vartheta}$ is the consistent estimate of ϑ already mentioned. The asymptotic power of the test is then

$$(2\pi)^{-1/2}\int_{\nu(\alpha)}^{\infty} \exp\{-\tfrac{1}{2}(t-\sigma_0(\vartheta)\tau)^2\}dt,$$

where $\tau=\lim_{n\to\infty}\xi_n\sqrt{n}$ and the expression is equal to α where $\tau=0$. *H. L. Seal* (New Haven, Conn.)

3057:

Ruben, Harold. On the distribution of the weighted difference of two independent Student variables. J. Roy. Statist. Soc. Ser. B **22** (1960), 188–194.

The d-statistic is defined as follows: $d_{f_1,f_2;\theta}=t_1 \sin\theta - t_2\cos\theta$, where t_1 and t_2 are independently distributed Student-t variables, with f_1 and f_2 degrees of freedom respectively.

"The distribution of the d-statistic is first obtained in integral form. This allows d to be interpreted as the ratio of two independent random variables, the numerator of which is a Student-t variable, and the denominator, a function of a Beta variable. Explicit forms for the distribution of d as finite or infinite series involving incomplete Beta function ratios and incomplete Gamma function ratios are derived for the special cases when (i) the degrees of freedom and weights of the component variables are equal and (ii) one of the components is normal." (From the author's summary)

I. Guttman (Montreal, Que.)

3058:

Machek, Josef. On a comparison of two normal populations. Apl. Mat. **5** (1960), 210–215. (Czech. Russian and English summaries)

Author's summary: "In this note an approximate test is given for the hypothesis that two normal populations with completely unknown means and variances have equal $100p\%$ and $100q\%$ quantiles (p and q are given numbers, p possibly different from q). Also an approximation to the quantile of the non-central t-distribution is given. The proposed significance test might be said to be a 'non-central analogue' of Welch's solution of the Behrens-Fisher problem."

3059:

Bartholomew, D. J. A test of homogeneity for ordered alternatives. II. Biometrika **46** (1959), 328–335.

Continuation and extension of Part I [Biometrika **46** (1959), 36–48; MR **21** #3067].

Benjamin Epstein (Palo Alto, Calif.)

3060:

Hooke, Robert. Use of randomization in the investigation of unknown functions. J. Amer. Statist. Assoc. **53** (1958), 176–186.

The selection of levels (and number of observations at each level) of factors in the design of an experiment may influence the variance and bias of an estimator of a parameter in a conceptual model. The selection of a numerical integration formula and the selection of the values (assumed to be measured without error) of the independent variable (or variables) may be interpreted within the framework of the statistical design of an experiment, especially when it is assumed that the dependent variable (or variables) is subject to observational errors. In this paper an interesting presentation is given for the problem of one-dimensional integration when the observational errors of the dependent variable are uncorrelated and of equal variance. The author presents the problem assuming that the values of the independent variable are selected at random within subintervals such that selection of an appropriate polynomial type integration formula corresponds to examining the variance of an estimator and such that the selection of values of the independent variable corresponds to examining the bias of an estimator.

M. Muller (New York)

3061:

Wormleighton, R. Some tests of permutation symmetry. Ann. Math. Statist. **30** (1959), 1005–1017.

Suppose $(x_{1\alpha}, \cdots, x_{k\alpha})$, $\alpha=1, \cdots, n$, are n independent k-dimensional random vectors such that $P(x_{i_1\alpha}<\cdots<x_{i_k\alpha}) = 1/k!$ for each permutation $(i_1, \cdots, i_k)$ of $(1, \cdots, k)$ and all α. Let $y_{ij\alpha}=1$ if $x_{i\alpha}>x_{j\alpha}$, $i\neq j$, and $y_{ij\alpha}=0$ otherwise. Let $\beta_{ij}=2n^{-1/2}\sum_{\alpha=1}^{n}(y_{ij\alpha}-\frac{1}{2})$, and let $\|\sigma_{ii',jj'}\|$ be the covariance matrix of the $\binom{k}{2}$-dimensional random vector $(\beta_{ij}, j>i=1, \cdots, k)$. The author computes the matrix $\|\sigma_{ii',jj'}\|$ and its inverse $\|\sigma^{ii',jj'}\|$ and shows that the limiting distribution of

$$Q_n = \sum_{i'>i=1}^{n}\sum_{j'>j=1}^{n}\sigma^{ii',jj'}\beta_{ii'}\beta_{jj'}$$

as $n\to\infty$ is chi-square with $\binom{k}{2}$ degrees of freedom. He points out that Q_n can be used as a test for the null hypothesis that $P(x_{i\alpha}>x_{j\alpha})=\frac{1}{2}$, $i\neq j$. The analysis is extended to the case where the probabilities $P(x_{i_1\alpha}<\cdots<x_{i_k\alpha})$ have arbitrary values for the $k!$ permutations $(i_1, \cdots, i_k)$ of $(1, \cdots, k)$. The author also extends his analysis of all possible pairs of the random vectors $(x_{1\alpha}, \cdots, x_{k\alpha})$, $\alpha=1, \cdots, n$, to one of all possible sets of m $(2\leq m\leq k)$ of these vectors by defining

$$y_{i_1i_2\cdots i_m\alpha} = 1 \quad \text{if} \quad x_{i_1\alpha} > \cdots > x_{i_m\alpha},$$
$$= 0 \quad \text{otherwise},$$

and

$$\beta_{i_1i_2\cdots i_m} = [np_{i_1i_2\cdots i_m}(1-p_{i_1i_2\cdots i_m})]^{-1/2}\sum_{\alpha=1}^{n}(y_{i_1i_2\cdots i_m\alpha}-p_{i_1i_2\cdots i_m}),$$

where $(i_1, i_2, \cdots, i_m)$ is a subset of the integers $(1, \cdots, k)$ and $p_{i_1i_2\cdots i_m}$ is the value of $P(x_{i_1\alpha}>\cdots>x_{i_m\alpha})$, which is constant for all α. The author points out that the $\beta_{i_1i_2\cdots i_m}$ satisfy $\binom{k}{m}$ independent homogeneous linear conditions, and indicates how a chi-square test for large n can be devised for testing the null hypothesis that the $P(x_{i_1\alpha}>\cdots>x_{i_m\alpha})$ has specified values $p_{i_1i_2\cdots i_m}$. The case $m=k$ is particularly simple and the chi-square test is given explicitly. Some discussion is given concerning the classes of alternative statistical hypotheses against which these tests are consistent.

S. S. Wilks (Princeton, N.J.)

3062a:

Dempster, A. P. A high dimensional two sample significance test. Ann. Math. Statist. **29** (1958), 995–1010.

3062b:

Dempster, A. P. A significance test for the separation of two highly multivariate small samples. Biometrics **16** (1960), 41–50.

Given are two p-variate samples of n_1 and n_2 items $(n_1+n_2=n)$ drawn from two Normal universes $N(u^{(1)}, \Sigma)$ and $N(u^{(2)}, \Sigma)$. Hotelling's statistic to test the equality

of the p-vectors of means $\mu^{(1)}$ and $\mu^{(2)}$ is $T^2=n^{-1}n_1n_2\times(\bar{\mathbf{y}}^{(1)}-\bar{\mathbf{y}}^{(2)})'\mathbf{S}^{-1}(\bar{\mathbf{y}}^{(1)}-\bar{\mathbf{y}}^{(2)})$, where the $\bar{\mathbf{y}}$'s are the sample means and $\mathbf{S}$ is the estimated covariance matrix [T. W. Anderson, *An introduction to multivariate statistical analysis*, Wiley, New York, 1958; MR **19**, 992]. This test is invariant under non-singular linear transformations of the observations, does not depend on Σ, but breaks down when $p>n-2$. For this case the author proposes the test statistic

$$F=(n-2)(\bar{\mathbf{y}}^{(1)}-\bar{\mathbf{y}}^{(2)})'(\bar{\mathbf{y}}^{(1)}-\bar{\mathbf{y}}^{(2)})/\sum_{i=3}^{n}\mathbf{Y}_i'\mathbf{Y}_i,$$

where, if $\mathbf{X}$ is the $n\times p$ matrix of the observations, $\mathbf{Y}=\mathbf{AX}$ is any non-singular orthogonal transformation such that if $\mathbf{Y}_i$ is the ith row of $\mathbf{Y}$, $\mathbf{Y}_1$ is the p-vector of grand means and $\mathbf{Y}_2$ is the p-vector of sample mean differences. Then F is non-invariant and depends on Σ. However it is shown in the first paper that its distribution can be approximated by Fisher's F-distribution with r and $n-2$ degrees of freedom where r is obtained from

$$(n-2)\ln\{(n-2)^{-1}\sum_{3}^{n}\mathbf{Y}_i'\mathbf{Y}_i\}-\sum_{3}^{n}\ln(\mathbf{Y}_i'\mathbf{Y}_i)\simeq\left\{\frac{1}{r}+\frac{1+(n-2)^{-1}}{3r^2}\right\}(n-3),$$

where the $\mathbf{Y}_i$ may now be taken as any arbitrary set of $n-2$ orthogonal Normal p-vectors which are also orthogonal to $\mathbf{Y}_1$ and $\mathbf{Y}_2$ defined above. The goodness of this and an improved approximation are illustrated in the second paper using 5 randomly chosen sets of $n-2=10$ unit orthogonal Normal 62-vectors. There are discussions of: (1) an alternative randomization procedure for the distribution of F; (2) the choice of the proper observational metric; (3) the sensitivity of the test; and (4) its asymptotic behavior when $p\geq r\to\infty$, n being fixed.

H. L. Seal (New Haven, Conn.)

3063:

Weibull, Martin. Moments of the difference between means in two samples from a finite population, applied in connection with a randomisation test. Skand. Aktuarietidskr. **1959**, 36–60.

Suppose $x_1,\cdots,x_N$ are measurements respectively on the N objects in a finite population. For any given permutation of the x's, say $(x_{i_1},\cdots,x_{i_N})$, let m_1 be the sum of the first n_1 x's and m_2 the mean of the next n_2 x's, $n_1+n_2\leq N$. Let $u=(m_1-m_2)/[\sigma(1/n_1+1/n_2)^{1/2}]$ where $\sigma^2=(N-1)^{-1}\sum_1^N(x_j-\mu)^2$ and $\mu=N^{-1}\sum_1^N x_i$. The author determines the first 8 moments and cumulants of u over all $N!$ permutations of $x_1,\cdots,x_N$ in terms of polykays, and fits the cumulative distribution function of u by a Charlier type A series to terms of order N^{-2}. He compares this fitted distribution with the actual distribution of u for various combinations of values of n_1 and n_2 ranging from 5 to 14, for values of N ranging from 10 to 20, and for (i) the case of ranks, that is, for $(x_1,\cdots,x_N)=(1,\cdots,N)$, (ii) the highly skewed case in which $(x_1,\cdots,x_i,\cdots,x_N)=\left(0,\cdots,\binom{i}{2},\cdots,\binom{N}{2}\right)$ and (iii) the case in which the cumulants K_r of the population are zero for $r=3,\cdots,8$.

For the case $n_1+n_2=N$ and $(x_1,\cdots,x_N)=(1,\cdots,N)$, the exact and approximate distributions of u follow from the work of Mann and Whitney.

S. S. Wilks (Princeton, N.J.)

3064:

Chatterjee, Shoutir Kishore. Some further results on the multinormal extension of Stein's two-sample procedure. Calcutta Statist. Assoc. Bull. **9**, 20–28 (1959).

Consider a p-dimensional normal distribution having means $(\mu_1,\cdots,\mu_p)$ and covariance matrix $\|\sigma_{ij}\|$. In a sample of size n_0 from this distribution let $\|s_{ij}\|$ be the usual unbiased estimator for $\|\sigma_{ij}\|$. For an arbitrary $z>0$ and positive definite matrix $\|a_{ij}\|$ let a second independent sample of size $n-n_0$, where

$$n=\max\{n_0+p^2,[z\sum_{i,j=1}^{p}a_{ij}s_{ij}]+1\}$$

be drawn from the same normal distribution. The author has shown in an earlier paper how to construct a random vector $(\xi_1,\cdots,\xi_p)$ from the two samples, and having mean value $(\mu_1,\cdots,\mu_p)$ such that

$$z\sum_{i,j=1}^{p}a_{ij}(\xi_i-\mu_i)(\xi_j-\mu_j)$$

has a chi-square distribution with p degrees of freedom, from which it follows that the random ellipsoid defined by

$$(1)\qquad \sum_{i,j=1}^{p}a_{ij}(\mu_i-\xi_i)(\mu_j-\xi_j)\leq\frac{\chi_\alpha^2}{z}$$

is a $100\alpha\%$ confidence ellipsoid of fixed size for the vector of "true" population means $(\mu_1,\cdots,\mu_p)$, where χ_α^2 is the 100α percentile of the chi-square distribution with p degrees of freedom.

In the present paper the author shows that if λ^* is the largest root of $|a_{ij}-\lambda s^{ij}|=0$, where $\|s^{ij}\|=\|s_{ij}\|^{-1}$, and if the value of n is taken as

$$\max\left\{n_0,\left[\frac{T_\alpha^2\lambda^*z}{\chi_\alpha^2}\right]+1\right\},$$

where T_α^2 is the 100α percentile of Hotelling's T^2 with (p,n_0-p) degrees of freedom, then

$$(2)\qquad n\sum_{i,j=1}^{p}s^{ij}(\mu_i-\bar{x}_i)(\mu_j-\bar{x}_j)$$

is a $100\alpha\%$ confidence region for $(\mu_1,\cdots,\mu_p)$ which lies inside that defined by (1), where $(\bar{x}_1,\cdots,\bar{x}_p)$ is the vector of the grand sample obtained by pooling the two samples.

S. S. Wilks (Princeton, N.J.)

3065:

Pfanzagl, J. Tests und Konfidenzintervalle für exponentielle Verteilungen und deren Anwendung auf einige diskrete Verteilungen. Metrika **3** (1960), 1–25. (English summary)

Let x_i, $i=1,2$, be independent random variables whose distributions belong to the exponential family. The densities of these distributions, each relative to some σ-finite measure, are given by $C_i(\eta_i)e^{\eta_i x}$, where the η_i are parameters. The author uses the fact [cf. E. L. Lehmann, *Testing statistical hypotheses*, Wiley, New York, 1959; MR **21** #6654] that the conditional distribution of x_1, given x_1+x_2, is an exponential distribution of the above, mentioned type with parameter $\eta_1-\eta_2$, for constructing a uniformly most powerful test for a two-sample problem with the hypothesis $\eta_1-\eta_2=0$ against a one-sided alternative. He generalizes this to several related k-sample problems. Using the terminology and results of a previous paper [Metrika **2** (1959), 11–45; MR **21** #943] he also

constructs for these problems locally most powerful tests. Finally he applies his results to a two-parametric subclass $\{P(\eta, \xi)\}$ of the exponential family having the following properties: For every fixed $\xi, P(\eta, \xi)$ is a complete set of measures. The convolution of $P(\eta, \xi_1)$ with $P(\eta, \xi_2)$ gives $P(\eta, \xi_1+\xi_2)$. If the independent random variables x_i have respective the distribution $P(\eta, \xi_i)$ $(i=1, 2)$, then x_1+x_2 is sufficient for η in the presence of the nuisance parameters ξ_1, ξ_2.

Specialising these results to the Binomial and Poisson distribution he obtains several known particular cases.

L. Schmetterer (Hamburg)

3066:

Williams, E. J. ★Regression analysis. A Wiley Publication in Applied Statistics. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1959. ix+214 pp. $7.50.

This book is written for research workers in the experimental sciences and is directed primarily to the problem of predicting Y from a knowledge of one or more X's. It is assumed that the reader has a good background in analysis of variance and similar statistical techniques. The excellent introductory chapter includes a discussion of model building, the role of significance tests in regression analysis and the uses of fiducial limits and multiple comparisons.

A number of chapters are devoted to traditional regression problems: linear regression with one or more X's; polynomial regression; estimation from regression equations; analysis of covariance; discriminant functions; functional relations. A large number of topics are treated in these chapters: methods of fitting regression lines to grouped data; transformations of data; allocation of X-values; selection of X-variables; computational procedures in multiple and polynomial regression; determining the degree of the polynomial to be used; estimation with linear restrictions on the regression coefficients; changes when X-variables are added or omitted; methods of correcting Y or X_1 for X_2; construction of fiducial limits (confidence limits to this reviewer) for parameters, expected values of Y and X, and ratios of parameters (e.g., as used in determining points of maximum); probability limits for predicted values (called tolerance limits by the author but confidence limits by this reviewer and others); uses of covariance to account for variation due to one X (environmental) and to strengthen relations between one X and other X's (explanatory); discriminant functions for more than two groups; use of functional relations, especially for calibration and selection indexes; controlled variables (Berkson case) and instrumental variables. The author also discusses a number of special topics, often neglected in books on regression analysis: regression equations requiring iteration (models nonlinear in some parameters); how to choose between regression formulas; the analysis of heterogeneous data (regressions not the same for different groups, certain types of correlated data, two-stage sampling); simultaneous equations.

The only serious defect is a tendency to present too much in such a small book. As a result many topics are not discussed in enough detail for the casual reader; however, sufficient material may be available to entice the interested reader to pursue these topics in more detail elsewhere. This reviewer would have appreciated a more detailed discussion of the following topics: the effect of increasing the number of independent variables on the probability limits for predicted values; selection of the proper error variance when duplicates are used; estimation of the zeros of a fitted curve; regression based on data acquired in multi-stage sampling; effect of non-standard conditions on usefulness of least-squares methods (correlated errors; non-normality, especially for discriminant analysis; model errors). In the section on comparison of regression equations from sets of correlated data, the discussion would have been improved if a model had been included. In an example involving a two-way classification, the author indicates how to test for differences of regressions for one of the classifications. Two possible models are: $E(Y_{ij})=\beta_0+\alpha_i+\gamma_j+\beta_j x_{ij}$; $E'(Y_{ij})=\beta_0'+\alpha_i'+\gamma_j'+\beta x_{ij}$.

A large number of excellent examples are included; over half of them are concerned with experiments involving wood and paper products.

The book is replete with pertinent comments on experimental and analytical procedures which should be taken to heart by every experimenter. This review would not be complete without citing a few: (1) A test of significance is seldom the ultimate objective of an experiment. (2) The idea of relationships among errorless quantities is useless, whereas the regression concept based on relationships with observations proves to be exceedingly useful. (3) If the regression line is not constant from group to group, there is no need to test for the existence of an over-all line. (4) Allocation (of X's) depends on the use to which data is to be put and on prior knowledge. (5) In multiple regression, it is often possible to array the X-variables in order of expected importance; computational procedures can take advantage of such ordering. (6) In fitting a polynomial function, all orthogonal polynomials up to that of highest degree fitted must be used. (7) The dependent variable must be a random variable; if both X and Y are random, use as the dependent variable the one to be predicted.

R. L. Anderson (Raleigh, N.C.)

3067:

Patterson, H. D.; Lipton, S. An investigation of Hartley's method for fitting an exponential curve. Biometrika **46** (1959), 281–292.

The authors investigate various procedures for estimating ρ in the exponential regression curve $E(y_x)=\alpha-\beta\rho^x$ $(0<\rho<1)$, $x=0, 1, \cdots, n-1$. The estimator

$$r = \sum_1^{n-1} w_x y_x \Big/ \sum_1^{n-1} w_x y_{x-1} = A_1/A_2, \quad \sum w_x = 0,$$

provides consistent estimates of ρ. Hartley's procedure [Biometrika **35** (1948), 32–45; MR **10**, 50] belongs to a general class of quadratic estimates, in which the weights (w_x) are linear functions of the y_x. A notation for such estimates with minimum variance when $\rho=\rho_0$ is $r(\rho_0, k/l)$, where $w_i=\sum d_{ij}(ky_{i-1}+ly_i)$, the d_{ij} being defined as certain functions of ρ_0. Hartley's estimate is equal to $r(1, 1)$.

Approximate variances and biases of the estimates, $r(1, k/l)$, are compared with those for least squares estimates, for selected values of $\theta=\rho^{n-1}$; $n=4(1)7, 9, 12, 20$; $k/l=0$, 1 and ∞. Large sample results are also available, and values of k/l are given for zero bias.

These quadratic estimates are also compared with "linear" estimates, $r(\rho_0)$, the latter having weights independent of the y's.

The authors state, "While Hartley's method generally leads to estimates having relatively high efficiencies and small biases, and can therefore be regarded as adequate for many purposes, the same conclusion does not hold for similar methods which are sometimes used in place of the original Hartley method." This reviewer wishes to add that research is needed on the impact of the existence of large experimental errors on the conclusions presented here, for example, on the adequacy of the approximations used in computing the biases and variances.

R. L. Anderson (Raleigh, N.C.)

3068:

Hartley, H. O. The efficiency of internal regression for the fitting of the exponential regression. Biometrika **46** (1959), 293–295.

This note compares the large sample results in the above article by Patterson and Lipton with some results obtained by a student of the author. The author comments on needed research in this area, especially when constancy of the error variance is in doubt and where the correct model is of the Markoff type or a combination of the exponential and Markoff.

R. L. Anderson (Raleigh, N.C.)

3069:

Linhart, H. A criterion for selecting variables in a regression analysis. Psychometrika **25** (1960), 45–58.

The k possible predictor variables and the predicted variable are assumed to have a $(k+1)$-dimensional normal distribution. The problem deals with the exclusion of some r of the predictor variables from the regression analysis. The basis of the decision for exclusion, a measure for the precision of prediction, is the value $E(l)$ where l is the length of the confidence interval of the predicted variable and where the expectation must be taken over all possible regression samples of size n and over all possible predictor sets. The theory is largely based on a previous paper by the author (which is summarized in an appendix to this paper) and the technique features the square root method of reduction.

P. S. Dwyer (Ann Arbor, Mich.)

3070:

Tikkiwal, B. D. On the theory of classical regression and double sampling estimation. J. Roy. Statist. Soc. Ser. B **22** (1960), 131–138.

Given a simple random sample, selected without replacement, of n from N, let $\bar{x}_n$, $\bar{y}_n$ be the sample means and $\bar{x}_N$, $\bar{y}_N$ the population means of variates x and y. Let β be the regression coefficient of y on x, assumed known. It is shown that the minimum-variance unbiased estimator of $\bar{y}_N$ which is linear in the sample values of x and y is $\hat{Y}_r=\bar{y}_n+\beta(\bar{x}_N-\bar{x}_n)$. The analogous estimator for double (i.e., two-phase) sampling is also shown to be a best linear unbiased estimator.

Unbiased estimators of variance for situations in which β has to be estimated from the sample are given for single- and two-phase sampling from a bivariate normal population and, with an extended definition of unbiasedness, for sampling from a finite population regarded as having been obtained by sampling from an underlying normal population.

J. Durbin (London)

3071:

Watterson, G. A. Linear estimation in censored samples from multivariate normal populations. Ann. Math. Statist. **30** (1959), 814–824.

Methods of linear unbiased estimation initiated by A. K. Gupta [Biometrika **39** (1952), 260–273; MR **14**, 487] for the censored univariate normal distribution, and by Sarhan and Greenberg [same Ann. **27** (1956), 427–451; J. Amer. Statist. Assoc. **52** (1957), 58–87; same Ann. **29** (1958), 79–105; MR **18**, 238; **19**, 331; **20** #340] for the censored univariate normal and exponential distributions, are generalized by the use of the following considerations.

A k-variable random vector is sampled n times. A numerical ordering for the observations on one variable induces a (nonnumerical) ordering for the observations on the additional $k-1$ variables. The result is called the ordered sample. Three types of censoring are distinguished. It is shown, among other things, for the bivariate case, and for two types of censoring, that "alternative" estimators of the Gupta type are generally (for most values of ρ_{12}^2) relatively more efficient than other linear unbiased estimators, and have high absolute efficiency with respect to maximum likelihood estimators, for the parameters which may be estimated.

{The term "vector of the ordered sample" needs some clarification; the terms "variable", "variate", and "observation" need more consistent usage; and the generalization from the bivariate case to the multivariate case is too scanty to be really clear and convincing.}

R. F. Tate (Seattle, Wash.)

3072:

Dugué, Daniel. Sur l'analyse de la variance à plusieurs dimensions (Extension de la loi d'Hotelling). Le calcul des probabilités et ses applications. Paris, 15–20 juillet 1958, pp. 81–87. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXVII. Centre National de la Recherche Scientifique, Paris, 1959. 196 pp.

Given a sample of $\{n_1, n_2, \cdots, n_p\}$ observations from p groups, with each observation consisting of measurements on each of k variables, the author uses the generalized residual and between groups sums of squares and products matrices to test the hypothesis that the mean vector $(k\times 1)$ is the same for all groups. Let $\Delta(S)$ be the determinant of the $(k\times k)$ matrix of residual sums of squares and products, $\Delta(s)$ for between groups variation, and $\Delta(\Sigma)$ for the total variation, where $\Delta(\Sigma)=\Delta(S+s)$. The testing statistic is $u=\Delta(s)/\Delta(\Sigma)$, which is shown to be the product of k independent random Beta variables. It is not clear to this reviewer how this result simplifies the calculation of significance levels of u.

R. L. Anderson (Raleigh, N.C.)

3073:

Anderson, Harry E., Jr.; Fruchter, Benjamin. Some multiple correlation and predictor selection methods. Psychometrika **25** (1960), 59–76.

As the title implies, this paper deals with multiple correlation methods and, in particular, with methods which feature selection of suitable predictors. The Doolittle method is discussed with the Wherry-Doolittle method of test selection and the so-called Summerfield-Lubin method of test selection. It is not surprising that

the authors find that these last two methods lead to essentially the same results, given the same decision rules, since Summerfield and Lubin feature the square root (Choleski) method of solving equations which is, in a sense, a modification of the Doolittle and other Gaussian elimination methods. "The Summerfield-Lubin method, because of its compactness and ease of computation, and because of the meaningfulness of the interim computational values, is recommended as a convenient least square method of multiple correlation and predictor selection."

P. S. Dwyer (Ann Arbor, Mich.)

3074:

Lord, Frederic M. Large-sample covariance analysis when the control variable is fallible. J. Amer. Statist. Assoc. **55** (1960), 307–321.

Analysis of covariance as applied to the model

$$y = \alpha_i + \beta\xi + \varepsilon \quad (i = 0, 1),$$

where $\alpha_1 - \alpha_0$ is the treatment difference under analysis, is extended to the case when the predictor variable ξ is subject to measurement error, say v. To avoid β becoming unidentifiable, as is the case on the customary assumption about measurement errors in both variables, it is assumed that each ξ is subject to two measurements with errors v that are independent and normal $(0, \sigma)$, where σ is the same for all v's in the two populations. The method is designed as a large sample procedure. It may result in either higher or lower significance levels than the conventional analysis of covariance methods, which ignore errors of measurement in ξ. The computation scheme is illustrated by data where the second measurement of ξ is fictitious.

H. Wold (Uppsala)

3075:

Tikkiwal, B. D.; Kabe, D. G. On the proof of the lemma used in deriving the distribution of the second order sample moments for a symmetric *p*-variate population. J. Karnatak Univ. **3** (1958), no. 1, 113–115.

The following lemma is proved. Let Y be a random column vector of n components, $n > 2$, having the density $f(y'y, By)$, where the given q by n matrix B is of rank q. Then the joint density of $U = Y'Y$, $V = BY$, is given by

$$\tfrac{1}{2}C(n-q)f(u, v)\begin{vmatrix} u & v' \\ v & BB' \end{vmatrix}^{\frac{1}{2}(n-q)-1} |BB'|^{-\frac{1}{2}(n-q-1)},$$

where $C(i) = (2\pi)^{i/2}(i/2)^{-1/2}$, $i \geq 2$.

L. A. Aroian (Los Angeles, Calif.)

3076:

Fisz, M. Some non-parametric tests for the *k*-sample problem. Colloq. Math. **7** (1959/60), 289–296.

This is an expository summary of methods and results for nonparametric tests based on functions of the distances between several sample cumulative distributions. Most of the material relates to large sample distribution theory under the null hypothesis, but the importance of ascertaining the distribution theory under alternative hypotheses is emphasized.

I. R. Savage (Cambridge, Mass.)

3077:

Sethuraman, J.; Sukhatme, B. V. Joint asymptotic distribution of *U*-statistics and order statistics. Sankhyā **21** (1959), 289–298.

Authors' summary: "It is shown under some mild restrictions that the joint distribution of a U-statistic (Hoeffding) and the a_nth order statistic tends to (i) the bivariate normal distribution if $a_n/n \to p$, $0 < p < 1$, (ii) the joint distribution of two independent variables, one of which is gamma and the other normal, in case $a_n \to$ constant or $n - a_n \to$ constant, (iii) the joint distribution of two independent normal variables if $a_n \to \infty$ such that $a_n/n \to 0$ or $n - a_n/n \to 0$. The above results are generalised to the case of several order statistics and several U-statistics. The generalisation to the case of several populations and generalized (Lehmann) U-statistics is also pointed out."

W. Hoeffding (Chapel Hill, N.C.)

3078:

Konijn, H. S. Positive and negative dependence of two random variables. Sankhyā **21** (1959), 269–280.

Author's summary: "Two random variables will be called completely positively dependent if there is an almost sure nondecreasing relation between them, and positively κ-dependent if their joint distribution is a mixture (with mixture coefficient κ) of the distribution of two completely positively dependent random variables and the distribution of two independent random variables with the same marginals. Similar definitions can be given for complete negative dependence and negative κ-dependence. The paper discusses properties of such variates and properties of the power of various tests for independence against such types of dependence."

W. Hoeffding (Chapel Hill, N.C.)

3079:

Gupta, Shanti S. Order statistics from the gamma distribution. Technometrics **2** (1960), 243–262.

Let $x_{(1)} \leqq x_{(2)} \leqq \cdots \leqq x_{(N)}$ be the order statistics of a sample of size N from the gamma distribution having probability density function

$$f_r(x) = \frac{x^{r-1}}{\Gamma(r)}e^{-x}, \quad x \geqq 0.$$

If $\mu_i'(k, N)$ denotes the ith moment of $x_{(k)}$ about the origin and if

$$\nu_i'(k, N) = (N!)^{-1}(k-1)!(N-k)!\Gamma(r)\mu_i'(k, N),$$

the author shows that these quantities satisfy the equation

$$\nu_i'(k, N-1) = \nu_i'(k, N) + \nu_i'(k+1, N), \quad k \leq N-1.$$

From this relation values of $\mu_i'(k, N)$ $(i = 1, 2, 3, 4)$ are tabulated for $r = 1, 2, 3, 4, 5$ and for k, N as follows: $1 \leq k \leq N$, $N = 1(1)10$, and for $k = 1$, $N = 11(1)15$. The moments about the origin, namely $\mu_i(k, N)$, are given for the same combinations of values of r, k, N.

The author also tabulates the modal values of the distributions of $x_{(1)}$ and of $x_{(N)}$ for $r = 1(1)10$ and $N = 2(1)10, 15, 20, 25, 50, 100$. Using the fact that the cumulative distribution function of $x_{(k)}$, say, $F_r(y; k, N)$, is given by

$$F_r(y; k, N) = I_{G_r(y)}(k, N-k+1),$$

where $G_r(y) = \int_0^y f_r(x)dx$, and $I_\alpha(p, q)$ is the incomplete Beta distribution, the author tabulates the α-percentage point of $x_{(k)}$ (the values of y_α for which $F_r(y_\alpha; k, N) = \alpha$)

for $\alpha = .01, .05, .10, .25, .50, .75, .90, .95, .99$, for $r = 1, 2, 3, 4, 5$, and for $1 \leq k \leq N$, $N = 1(1)10$, and for $k = N = 25, 50, 100, 200, 500, 1000$.

Illustrative applications to life-testing and reliability problems are given. *S. S. Wilks* (Princeton, N.J.)

3080:

Daniel, Cuthbert. Parallel fractional replicates. Technometrics **2** (1960), 263–268.

Consider a multi-factor experimental situation where there are two response variables. By taking account of a priori information on which factors influence the response variables, the author suggests several 2^{p-q} fractional factorial designs which may be useful for analyzing both response variables in the same experiment.

M. Zelen (College Park, Md.)

3081:

Taguchi, Genichi. Tables of orthogonal arrays and linear graphs. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **7** (1960), 1–52.

Various types of linear graphs which can be used for designing experiments using orthogonal arrays are collected in this table. No expository material whatever is included with the tables or the graphs.

R. G. Stanton (Waterloo, Ont.)

3082:

Dykstra, O., Jr. Partial duplication of response surface designs. Technometrics **2** (1960), 185–195.

The author presents experimental plans for central composite second order response surface experimental designs with replication of some of the experimental points. This is a continuation of the author's work in Technometrics **1** (1959), 63–76 [MR **21** #5261]. The partial replication is carried out either on (i) a fraction of the points in the hypercube or (ii) the "star" portion of the design. *M. Zelen* (College Park, Md.)

3083:

Trybula, S. On some loss functions. Colloq. Math. **7** (1959/60), 297–305.

For estimating the parameters $p_1, \cdots, p_k$ of the multinomial distribution, the unbiased estimates are shown to be the unique minimax out of all estimates $f_1, \cdots, f_k$ for which $f_1 + \cdots + f_k = 1$, when the loss function, $L(f, p) = \sum_1^k (f_i - p_i)^2/p_i$, is used. The uniform distribution on the parameter space is shown to be least favorable. A similar result is proved for the multivariate hypergeometric distribution. *T. S. Ferguson* (Los Angeles, Calif.)

3084:

Wetherill, G. B. The most econcmical sequential sampling scheme for inspection by variables. J. Roy. Statist. Soc. Ser. B **21** (1959), 400–408.

Approximations due to E. S. Page [same J. **16** (1954), 136–139; MR **16**, 498] are applied to the optimal (Waldian) sequential two-decision procedure involving a two-point prior distribution.

W. Hoeffding (Chapel Hill, N.C.)

3085:

Cusimano, Giovanni. Problemi di interferenza delle macchine nella teoria statistica delle file d'attesa. Statistica. Bologna **19** (1959), 361–414. (1 insert)

Expository article containing many references to the literature of the problem of machine interference.

L. Schmetterer (Hamburg)

3086:

Nanjamma, N. S.; Murthy, M. N.; Sethi, V. K. Some sampling systems providing unbiased ratio estimators. Sankhyā **21** (1959), 299–314.

In the years just before 1940 the advantages of biased estimators in the reduction of mean square error in survey sampling were appreciated. About 1950 the possibility of removing the bias by the manner of choosing the sample was recognized. The authors give a relatively complete treatment, applicable to many common sampling designs, of procedures for selecting samples so that estimates based on ratio estimates shall be unbiased, consisting essentially in selecting a first unit with probability proportionate to the size of its denominator variate and the remaining units according to the original sampling design. A generalized procedure for obtaining unbiased estimates which should be of wide application is outlined. [Bibliography: Des Raj, J. Indian Agric. Statist. **6** (1954), 127–138; MR **18**, 682; H. Midzuno, Ann. Inst. Statist. Math. Tokyo **1** (1950), 149–156; **3** (1952), 99–107; MR **12**, 36; **14**, 390; D. B. Lahiri, Bull. Inst. Internat. Statist. **33** (1955), 133–140; A. R. Sen, Econometrica **20** (1954), 103.] *C. J. Maloney* (Frederick, Md.)

3087:

Hannan, E. J. The estimation of seasonal variation. Austral. J. Statist. **2** (1960), 1–15.

The author makes the conventional assumption that the variable can be represented as the sum of trend, seasonal and random terms, and considers the calculation of the seasonal component by moving averages or by the fitting of regressions. He emphasises the use of Fourier methods in the evaluation of moving average formulae, and gives a detailed discussion of the estimation of variances of regression coefficients.

P. Whittle (Cambridge, England)

NUMERICAL METHODS

See also A2514, A2727, A2781, 3128, 3195, 3382, 3383, 3596, 3597.

3088:

de Freitas, A. César. Numerical analysis and automatic numerical computation. Ciência. Lisboa. No. 15/16 (1958/59), 29–32. (Portuguese)

An expository lecture in semi-popular style.

3089:

Langer, Rudolph E. (Editor). ★Frontiers of numerical mathematics. A symposium conducted by the Mathematics Research Center, United States Army and the National Bureau of Standards at the University of

Wisconsin, Madison, Wisconsin, October 30 and 31, 1959. University of Wisconsin Press, Madison, Wis., 1960. xi+132 pp. \$3.50.

The eight articles will be reviewed individually.

3090:

Schröder, Johann. Error estimates for boundary value problems using fixed-point theorems. Boundary problems in differential equations, pp. 85–96. Univ. of Wisconsin Press, Madison, 1960.

Consider a boundary value problem of the form $M[u]=f_0(x, u)$ on B, $U_i[u]=f_i(x, u)$ on Γ_i $(i=1, 2, \cdots)$ where B is a domain of the p-dimensional Euclidean space, Γ_i are parts of the boundary of B and M, U_i are linear differential operators. Denoting a solution of this problem by u^* and an approximation of u^* by u_0, the author derives error-estimates of the form $|u^*(x)-u_0(x)| \leq \eta(x)$. The central theorem about these estimates is essentially a somewhat specialized and simpler version of more general theorems derived by the author in several other papers [J. Schröder, Arch. Rational Mech. Anal. **3** (1959), 219–228; **4** (1959), 177–192; MR **21** #7596; **22** #319]. In simplified terms the common idea of all these theorems is to write the given problem as an operator equation $u=Tu$ and to obtain subsets K of a space R containing both domain and range of T such that $TK \subset K \subset R$. Then assumptions are made which assure that a fixpoint-theorem is true for T in the sets K under consideration, i.e., that $u^*=Tu^* \in TK \subset K$. For a suitable choice of K this inclusion constitutes a numerically usable estimate for u^*. In this paper one takes $K=\{u \in R, |u-u_0| \leq \eta\}$ where $\eta(x)$ is a solution of a certain "comparison-problem". The author then proceeds to outline a technique for the numerical evaluation of u_0 and η using difference methods and linear approximations. Finally, several examples are given to illustrate the results.

W. C. Rheinboldt (Syracuse, N.Y.)

3091:

Arsac, Jacques. Approximation de fonctions numériques au moyen de translatées d'une fonction étalon. C. R. Acad. Sci. Paris **250** (1960), 445–447.

Suite d'une note antérieure de l'auteur [mêmes C.R. **250** (1960), 278–280; MR **22** #1059]. Application des résultats trouvés aux interpolations de Lagrange, Bessel et Stirling. Définition de nouveaux types d'interpolation.

J. Kuntzmann (Grenoble)

3092:

Pokrovskiĭ, V. L. On a class of polynomials with extremal properties. Mat. Sb. (N.S.) **48 (90)** (1959), 257–276. (Russian)

Soit $P_n(x)$ le polynôme algébrique d'ordre n, $B_n(x)=P_n(x)(1+x^2)^{-n/2}$, $-a<c<a$, $(\alpha, \beta) \subset (-a, a)$. Envisageons parmi les fonctions B_n celles qui sont caractérisées par les propriétés suivantes: Sur $(-a, a)$ $B_n(x)$ prend la valeur maximale, égale à R, au point $x=c$. $|B_n(\alpha)|=|B_n(\beta)|=1$, $|B_n(x)|<1$ pour $x \in (-a, \alpha)$, (β, a), tandis que $|B_n(x)|>1$ pour $x \in (\alpha, \beta)$. Soit $(B_n)=B_n(c, R, a)$ la classe des fonctions jouissant de ces propriétés, et $B_n^0(x) \in (B_n)$ celle qui minimise l'écart $(\beta-\alpha)$. On considère les points caractéristiques, c'est à dire, les points consécutifs où $B_n(x)=1$, ou bien $B_n(x)=-1$, et l'on fait une classification de (B_n) d'après la répartition de ces points, on donne des propriétés de B_n^0 et on démontre que si le nombre de ces points dépasse $(n-1)$ il existe une et une seule fonction $B_n^0(x)$. L'auteur remarque dans l'introduction que ce problème mathématique a son origine dans la construction des antennes de T.S.F.

M. Tomić (Belgrade)

3093:

Scheffler, D.; Ondrejka, R. The numerical evaluation of the eighteenth perfect number. Math. Comput. **14** (1960), 199–200.

The decimal representation of the perfect number $2^{3216}(2^{3217}-1)$ corresponding to the Mersenne prime $2^{3217}-1$ is given explicitly. The number contains 1937 decimal digits and was computed on an IBM 709 on 2 separate occasions, each computation taking approximately 5 minutes.

M. Newman (Washington, D.C.)

3094:

Fieldhouse, M. The saddle-points of the confluent hypergeometric function $M(a, b; x)$. Proc. Cambridge Philos. Soc. **56** (1960), 148–153.

The zero-surface of the confluent hypergeometric function $M(a, b; x)=0$ in a Euclidean x-a-b-space has infinitely many saddle points. After discussing shortly the theory and the computational process the author gives a table of about 100 saddle points for $x>0$, $-21<b<a<0$. Values are given to 5 decimal places.

F. Stallman (Washington, D.C.)

3095:

Lang, H. A.; Stevens, D. F. On the evaluation of certain complex elliptic integrals. Math. Comput. **14** (1960), 195–199.

Authors start from formulas for the sum and difference of complex conjugate elliptic integrals of the third kind found in Hoüel [G. J. Hoüel, *Recueil de formules et de tables numeriques*, Gauthier-Villars, Paris, 1901], and in the process of correcting them also correct, modify and simplify corresponding formulas found in Byrd and Friedman, *Handbook of elliptic integrals for engineers and physicists* [Springer, Berlin, 1954; MR **15**, 702].

M. D. Friedman (Needham Heights, Mass.)

3096:

de Witte, Leendert; Fournier, Kenneth P. Evaluation of integrals involving combinations of Bessel functions and circular functions. J. Assoc. Comput. Mach. **5** (1958), 119–126.

The authors outline a method for evaluation of infinite integrals containing combinations of Bessel functions and circular functions, in which the non-circular part of the integrand is fitted piecewise by combinations of polynomials, exponentials and logarithms.

The corresponding component integrations may then be carried out, and tables made from which the original integrals are readily derived.

J. C. P. Miller (Cambridge, England)

3097:

Alway, G. G. Multhopp's influence functions and their automatic computation. Quart. J. Mech. Appl. Math. **13** (1960), 112–118.

There are nine important Multhopp influence functions, but they are not all linearly independent. Three may be taken as fundamental and the remainder written in terms of them. These three may, in turn, be expressed as functions of three complete elliptic integrals. An iterative method is given for the evaluation of these integrals and the conveyance is stated to be rapid for most values of the two independent variables X and Y. However, to ensure accuracy in the final results, double length precision is necessary in certain parts of the calculation. Two appendices are included; these give the derivations of the formulae used in the main part of the paper.

G. N. Lance (Winfrith, Dorset)

3098:

Markowitz, Harry M. The elimination form of the inverse and its application to linear programming. Management Sci. **3** (1957), 255–269.

In solving linear equations by elimination of variables and back substitution, it is possible to choose pivot elements in more than one way. For a matrix with many zeroes, it is reasonable to attempt to choose the pivots so as to keep the number of non-zero elements which must be handled small. In part 1 of this paper, the author discusses some proposals for doing this and reports computational experience with one of the proposals.

In part 2, the author solves (under two sets of assumptions) the question of when a linear programmer should "reinvert" if the "product form of the inverse" version of the simplex method is used. The answer displays how time saved by using the suggestion of part 1 contributes to time saved on the whole linear programming calculation.

A. J. Hoffman (New York)

3099:

Bertram, G. Beziehungen zwischen Defektabschätzungen und "Linear programming" bei linearen Gleichungssystemen. Z. Angew. Math. Mech. **40** (1960), 373.

3100:

Pall, Gordon; Seiden, Esther. A problem in abelian groups, with application to the transposition of a matrix on an electronic computer. Math. Comput. **14** (1960), 189–192.

The sub-routine for the transposition of a $m\times n$ matrix takes very little memory space and very little time if it is done as follows: The element in the position $a=mj+i$ must be interchanged with the element in position $b=ni+j$. Clearly $na\equiv b \bmod N$ $(=nm-1)$. Then we take the element in position $b=mj'+i'$ and interchange it with the element in position $ni'+j'\equiv nb\equiv n^2a \bmod N$. This leads to a cycle of simple interchanges of length r where n belongs to $r \bmod [N/gcd(a, N)]$. The authors give an explicit construction for a set of so-called leaders a which will generate cycles $n^za \bmod N$ to exhaust the matrix. At present the authors imagine the leaders to be calculated ahead of time on paper and put into the machine. The problem is not without significance in the construction of a complete set of cosets of a subgroup of a general abelian group.

H. Cohn (Tucson, Ariz.)

3101:

Swift, George. A comment on matrix inversion by partition. SIAM Rev. **2** (1960), 132–133.

The comments are elementary observations on the case of a matrix that is blockwise triangular.

A. S. Householder (Oak Ridge, Tenn.)

3102:

Dück, W. Fehlerabschätzungen für das Iterationsverfahren von Schulz zur Bestimmung der Inversen einer Matrix. Z. Angew. Math. Mech. **40** (1960), 192–194.

3103:

Uhlig, Joachim. Ein Iterationsverfahren für ein inverses Eigenwertproblem endlicher Matrizen. Z. Angew. Math. Mech. **40** (1960), 123–125.

The problem is, given a matrix A, to find a diagonal matrix D such that the characteristic roots of AD are prescribed. The method proposed is as follows: If $\mu_1, \cdots, \mu_n$ are the prescribed roots, find $d_i^{(1)}$ so that $\det(AD_i^{(1)}-\mu_i I)=0$, where $D_i^{(1)}$ differs from I only by having $d_i^{(1)}$ in the ith diagonal position. Then let $D^{(1)}=\operatorname{diag}(d_1^{(1)}, \cdots, d_n^{(1)})$, $A^{(1)}=AD^{(1)}$. Repeat with $A^{(1)}$ in place of A. Convergence is not proved.

A. S. Householder (Oak Ridge, Tenn.)

3104:

Dimsdale, B. Characteristic roots of a matric polynomial. J. Soc. Indust. Appl. Math. **8** (1960), 218–223.

The problem of finding the zeros of the polynomial

$$\det(A_n\lambda^n+A_{n-1}\lambda^{n-1}+\cdots+A_0),$$

where the A's are matrices of order m, can be easily transformed into the problem of finding the zeros of a matrix of order mn whose elements are linear in λ. If A_n or A_0 is not singular it is again a simple matter to transform the linear problem to the standard characteristic value problem. This paper describes an algorithm for the reduction to the standard form when A_n and A_0 are both singular.

There are several misprints in the paper. In the equation defining U on page 218 replace 0 in the last column by A_{n-2}, and $-\lambda I$ in the fourth column by 0. In the equation defining U_7 on page 221 interchange E_{12} and E_{21}.

B. A. Chartres (Sydney)

3105:

Mikeladze, Š. E. Some iterations of higher order. Soobšč. Akad. Nauk Gruzin. SSR **22** (1959), 257–264. (Russian)

For solving the scalar equation $f(z)=0$ by means of an iteration $z_n=\phi(z_{n-1})$, the author proposes a form

$$\phi(z) = z-\sum_1^m a_k\omega_k(z),$$

$$\omega_k = f(z)/f'(z+\beta_{k-1}\omega_{k-1}),$$

with appropriate choice of scalars a_k and β_k. This suggests a generalization to the case of f and z vectors in a finite-dimensional space in that the quotient f/f' becomes $(f')^{-1}f$, where f' is the Jacobian matrix.

A. S. Householder (Oak Ridge, Tenn.)

3106:

Longman, I. M. On the utility of Newton's method for computing complex roots of equations. Math. Comput. **14** (1960), 187–189.

The author gives two examples of the use of Newton's method for the solution of equations with complex roots.

B. A. Galler (Ann Arbor, Mich.)

3107:

Mack, C. Routh test function methods for the numerical solution of polynomial equations. Quart. J. Mech. Appl. Math. **12** (1959), 365–378.

By applying Routh's algorithm for determining the number of zeros of a polynomial $f(z)$ having positive real parts, the author evolves an iterative method for computing the real part of a zero of $f(z)$, which always converges. Subsequently, the imaginary part may be obtained trivially from the computations.

The second method given in the paper is the determination of a real quadratic factor z^2-az+b by taking arbitrary values of a, and testing the remainder $R_1(b)z+R_0(b)$ which results on dividing $f(z)$ by z^2-az+b, to ascertain whether or not the polynomials $R_0(b)$ and $R_1(b)$ have a common zero. Correct values of a are then determined iteratively by application of Routh's algorithm.

Numerical examples of the two methods are included. There is a misprint in the solution given for a quartic equation.

{Reviewer's note: No indication is given of whether the methods have been applied to high-degree polynomials; the effects of ill-conditioning, which are of paramount computational importance [J. H. Wilkinson, Numer. Math. **1** (1959), 150–180; MR **22** #321], are barely touched upon. Moreover, the methods seem somewhat laborious for low-degree polynomials. Using the method described in *Modern computing methods* [H.M.S.O., London, 1957; MR **19**, 579], the reviewer has computed by hand the solution of the given quartic equation in less time than the author claims for his methods.}

F. W. J. Olver (Teddington)

3108:

Albrecht, Julius. Über die Abrundungsfehler bei der Iteration für $y=\sqrt[p]{x}$. Z. Angew. Math. Mech. **40** (1960), 191.

3109:

Akkerman, R. B. Quadrature formulas of the type of Markov formulas. Trudy Mat. Inst. Steklov. **53** (1959), 5–15. (Russian)

The author considers quadrature formulae of the form

$$\int_a^1 x^p f(x)dx \approx A_0 f(a)+B_0 f'(a)+A_{n+1}f(1)+\sum_{k=1}^{n} A_k f(x_k)$$

in which A_k, x_k $(k=1,\cdots,n)$ are chosen to maximize the highest degree m of polynomials for which the formula is exact subject to certain restrictions. He gives coefficients and abscissae to 8 decimal places for $n=1,\cdots,N$ for the cases in which a, p, m, N have, respectively, sets of values as follows: (i) -1, 0, $2n+1$, 8, with $B_0=0$; (ii) 0, 1, $2n+1$, 4, with $B_0=0$; (iii) 0, 2, $2n+1$, 4, with $B_0=0$; (iv) 0, 0, $4n+2$, 8, with $B_0=0$ and $f(x)$ even; (v) 0, 0, $4n+3$, 8, with $A_0=0$ and $f(x)$ odd; (vi) -1, 0, $2n$, 5, with $A_0=B_0=0$; (vii) 0, $\frac{1}{2}$, $2n$, 5, with $B_0=A_{n+1}=0$; (viii) 0, 2, $4n$, 5, with $B_0=A_{n+1}=0$ and $f(x)$ even; (ix) 0, 0, $4n+1$, 5, with $A_0=A_{n+1}=0$ and $f(x)$ odd. He combines some of these results with the usual quadrature formula with equidistant abscissae for a periodic function, and so obtains coefficients and abscissae to 8 decimal places for formulae (x) for $\iint f(x,y)dxdy$ over the disc $x^2+y^2\leq 1$ with 5 [21] ordinates, exact if f is of degree at most 5 [9], [cf. L. A. Lyusternik and V. A. Ditkin, Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1163–1168; MR **10**, 484], and (xi) for $\iiint f(x,y,z)dxdydz$ over the sphere $x^2+y^2+z^2\leq 1$ with 16 [91] ordinates, exact if f is of degree at most 4 [8].

H. P. Mulholland (Exeter)

3110:

Wetterling, W. Zum Einschliessungssatz von Kryloff-Bogoliubov für allgemeine Eigenwertaufgaben bei gewöhnlichen Differentialgleichungen. Numer. Math. **2** (1960), 18–21.

Verf. beweist einen Einschliessungssatz für selbstadjungierte Eigenwertaufgaben bei gewöhnlichen Differentialgleichungen, welche nach Kamke [Math. Z. **46** (1940), 231–250; MR **2**, 52] im engeren Sinne definit heissen, mit der Differentialgleichung $M[y]=\lambda N[y]$ $(a\leq x\leq b)$ und den Randbedingungen $U_\mu[y]=0$ $(\mu=1, 2, \cdots, m)$. Seien u_0, u_1 und u_2 drei Vergleichsfunktionen mit $M[u_k]-t\cdot N[u_k]=N[u_{k-1}]$, wobei das Intervall $[0, t]$ keinen Eigenwert enthält, so liegt zwischen

$$t+\frac{a_2}{a_3}-\left(\frac{a_1}{a_3}-\left(\frac{a_2}{a_3}\right)^2\right)^{1/2} \quad\text{und}\quad t+\frac{a_2}{a_3}+\left(\frac{a_1}{a_3}-\left(\frac{a_2}{a_3}\right)^2\right)^{1/2}$$

mindestens ein Eigenwert $(a_k=\int_a^b u_1 N[u_{k-1}]dx)$. Im Beweis wird der Satz über die Entwicklung von Vergleichsfunktionen nach Eigenfunktionen nicht benutzt. Numerische Beispiele: $-y''=\lambda\times y$, $y(-1)=y(1)=0$ und $y^{\mathrm{iv}}=\lambda(y+cy'')$, $y(0)=y(1)=y''(0)=y''(1)=0$.

J. Schröder (Hamburg)

3111:

Lees, Milton. von Neumann difference approximation to hyperbolic equations. Pacific J. Math. **10** (1960), 213–222.

Let v_x, $v_{\bar{x}}$ and $v_{\hat{x}}$ represent, respectively, the forward, backward and centered difference quotients. Then two extensions of the von Neumann difference equations to the generalized wave equation are:

$$(*)\quad v_{t\bar{t}} = a(x,t)v_{x\bar{x}}+\gamma(x,t)\alpha(\Delta t)^2 v_{x\bar{x}t\bar{t}}+b(x,t)v_{\hat{x}}+c(x,t)v_{\hat{t}}+d(x,t)v+e(x,t),$$

where (1) $\gamma(x,t)\equiv 1$ or (2) $\gamma(x,t)\equiv a(x,t)$. It is proven, by extension of the Friedrichs and Lewy energy inequalities to difference inequalities, that (*) is an unconditionally stable difference scheme (subject to appropriate initial and boundary conditions) in case (1) if $|4\alpha-a(x,t)|\geq\varepsilon>0$ and in case (2) if $4\alpha-1>0$. These results imply the solvability of the corresponding linear systems and the pointwise convergence as $\Delta t\to 0$ and $\Delta x\to 0$ of the difference solution to the exact solution of the continuous analogue of (*). This convergence is $O(\Delta x^2+\Delta t^2)$ under assumed regularity conditions of the exact solution.

H. B. Keller (New York)

3112:

Fox, P.; Ralston, A. On the numerical solution of the equations for spherical waves of finite amplitude. I. J. Math. Phys. **36** (1958), 313–328.

This paper presents a numerical calculation of a spherical expansion wave resulting from an initial compressed sphere of gas. The ratio of density at the center to ambient is 3:1, and it falls off exponentially as function of radius. The results of the calculations presented here show that at first there is an outgoing spherical wave whose amplitude, because of the effects of divergence, decrease. The initially compressed inner region overexpands, the density at the center falling slightly below ambient. Thereupon a secondary compression wave, travelling inward, is formed which compresses the center to slightly above ambient density. By this time conditions are near equilibrium and so the calculation (performed on the "Whirlwind") was stopped. At no time during the calculation was there any indication of the formation of a shock wave.

The numerical method used is integration along the characteristics, with iteration at each step to assume proper centering. An ingenious special extrapolation and iteration technique was devised to deal with the singularity at $r=0$.

The accuracy of the calculations is checked by comparing the results of two calculations performed with different mesh sizes, and by the conservation of mass inside Lagrangean spheres. Excellent accuracy is indicated.

An earlier calculation (1941) by Unwin, using a different method, has indicated a very strong secondary compression wave. In an appendix the authors point out that Unwin's difference scheme has an unacceptably large truncation error, due to his failure to secure higher accuracy by iteration. Was kommt noch dazu, the calculations published by Unwin show a gross violation of the conservation of mass inside Lagrangean spheres. In view of these facts, one is forced to accept the calculations of Fox and Ralston as the correct one. For additional confirmation see the paper of Roberts reviewed below.

P. Lax (New York)

3113:

Roberts, Leonard. On the numerical solution of the equations for spherical waves of finite applitude. II. J. Math. Phys. **36** (1958), 329–337.

This paper presents a numerical solution using an explicit difference scheme proposed by the reviewer, based on the Euler equations of flow, of a spherical flow problem treated by Fox and Ralston in the paper reviewed above by the method of characteristics. The results of the calculations are in good agreement with those of Fox and Ralston [see preceding review].

The distinctive features of these calculations are (1) the modification of the equations to take care of the singularity at $r=0$, (2) the extrapolation procedure from two calculations performed with different values of Δt but the same value of $\Delta t/\Delta x$ to obtain a result with error $O(\Delta^2)$; this simple technique turns out to be remarkably effective.

P. Lax (New York)

3114:

Korobov, N. M. Approximate solution of integral equations. Dokl. Akad. Nauk SSSR **128** (1959), 235–238. (Russian)

The author considers the multiple integral equation

$$\varphi(P) = \lambda \int_{G_s} K(P, Q)\varphi(Q)dQ + f(P)$$

where P and Q are points in the unit cube G_s in s-dimensional space; the Fourier coefficients of the function f and kernel K are assumed to satisfy certain order conditions, and it is supposed that the Fredholm denominator $D(\lambda)$ is not zero. Let M_i denote the point with coordinates $\{i^r/p\}$ (fractional part), where $1 \le r \le s$ and $1 \le i \le p$, p being a prime greater than s. Then it is shown that a solution of the integral equation can be expressed as

$$\varphi(M_i) = \bar{\varphi}(M_i) + O(1/\sqrt{p}) \quad (1 \le i \le p),$$

where $\bar{\varphi}(M_i)$ satisfies the system of equations

$$\bar{\varphi}(M_j) = \frac{\lambda}{p} \sum_{i=1}^{p} K(M_j, M_i)\bar{\varphi}(M_i) + f(M_j) \quad (1 \le j \le p).$$

Two further approximate results of a more complicated type are proved.

R. A. Rankin (Glasgow)

3115:

Šahov, Yu. N. Approximate solution of second kind Volterra equation by means of iterations. Dokl. Akad. Nauk SSSR **128** (1959), 1136–1139. (Russian)

Number-theoretic methods due to Korobov for the approximate evaluation of repeated integrals [N. M. Korobov, same Dokl. **115** (1957), 1062–1065; **124** (1959), 1207–1210; MR **20** #5169; **21** #2848] are applied to obtain an approximate solution of the Volterra integral equation of the second kind. These methods were previously applied by Korobov [see preceding review] in obtaining an approximate solution of Fredholm's equation of the second kind.

J. F. Heyda (Cincinnati, Ohio)

3116:

Fisher, Michael E. Proposed methods for the analog solution of Fredholm's integral equation. J. Assoc. Comput. Mach. **5** (1958), 357–369.

The author shows that Wellman's modification of the Neumann iterative process for solving Fredholm's integral equation not only fails to converge for large $|\lambda|$ but also that Wellman's suggestion that suitable choice of a zero'th order approximation can produce convergence is also false. He suggests that a universally applicable method is obtained by suitable transformation of the kernel. This transformation can only be obtained in an empirical manner using a special type of high speed analogue computer but the resemblance of the process to that of relaxation suggests that human insight would be important if the method is to be effective. It does not appear that the method has been tested in practice.

A. D. Booth (London)

3117:

Schubert, Hermann; Haussner, Robert; Erlebach, Joachim. ★Vierstellige Tafeln und Gegentafeln. 3., neubearbeitete Aufl. Sammlung Göschen Bd. 81. Walter de Gruyter & Co., Berlin, 1960. 156 pp. (1 insert) DM 3.60.

Sixteen numerical tables are given, including Briggsian and natural logarithms, addition and subtraction logarithms, trigonometric functions and mortality tables.

3118:

Falk, S. Zur Tabulierung von Polynomen. Z. Angew. Math. Mech. **40** (1960), 275.

3119:

Hof, Hans. ★Ten place natural trigonometric tables: Sine-tangent, 0 to 90 degrees. Professional Supply Co., Jenkintown, Pa., 1959. iv + 1084 pp.

The interval in these tables of sines and tangents is 1 second of arc. Ten decimals are given throughout for the sine, and nine, ten or sometimes eleven significant figures for the tangent. One first difference is printed, which for the sine is the average of the first difference in each block of ten entries, while for the tangent the value, after about 15°, is the actual difference between two successive entries. The error of the printed values is stated to be about one unit in the last figure. The tables are useful, but there is no comment about the accuracy of interpolation, and the difficulty of handling such a large book is increased by the necessity for rotating it through a right angle!

L. Fox (Oxford)

3120:

Ryshik, I. M.; Gradstein, I. S. ★Summen-, Produkt- und Integral-tafeln. (Title and text also in English.) VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. xxiii + 438 pp. DM 56.00.

An English and German translation of the third Russian edition [GITTL, Moscow-Leningrad, 1951], noted, with table of contents, in MR **14**, 643. The errata in that edition have been incorporated in the text, misprints corrected and some supplementary remarks, particularly about Laplace transforms, added in an appendix. Also, an extensive bibliography has been added to the original.

COMPUTING MACHINES

See also 3093, 3100, 3116, 3382, 3383, 3491, 3505, 3596, 3597, 3630, 3634.

3121:

Alt, Franz L.; Booth, A. D.; Meagher, R. E. (Editors). ★Advances in computers. Vol. I. Academic Press, New York-London, 1960. x + 316 pp. $10.00.

The six articles in this first volume of a new series will be reviewed individually.

3122:

Deprit, A. Les machines à calculer digitales en Grande-Bretagne. Rev. Questions Sci. (5) **21** (1960), 325–345.

A clearly written account of the historical development of computing machines in Great Britain from the time of Charles Babbage to the present day. The paper is purely historical and contains no new ideas.

A. D. Booth (London)

3123:

Tarján, R. On the instrumentation of logical problems. Acta Tech. Acad. Sci. Hungar. **27** (1959), 371–383. (German, French and Russian summaries)

After a discussion of plausible, i.e., inductive, reasoning the author considers nets constructed from the following type of neurons: They fire when the algebraic sum of the excitating and inhibiting stimuli over a time interval exceed a threshold; their threshold rises to infinity immediately after firing and then monotonically decreases to normal.

S. Ginsburg (Santa Monica, Calif.)

3124:

Reitman, Walter R. Computers in behavioral science. Heuristic programs, computer simulation, and higher mental processes. Behavioral Sci. **4** (1959), 330–335.

A general discussion of heuristic programming, especially of the work of Newell, Shaw and Simon.

J. McCarthy (Cambridge, Mass.)

3125:

Edmonds, A. R. The generation of pseudo-random numbers on electronic digital computers. Comput. J. **2** (1959/60), 181–185.

Several well written expository papers (this one included) have appeared recently which review procedures for generating sequences of pseudo-random numbers on digital computers. This paper also contains mention of some statistical tests of randomness that might be applied to such sequences. A brief account is given of pseudo-random number subroutines which have been written for the Pegasus and Mercury computers.

M. Muller (New York)

3126:

Poyen, J. AP.3—Autoprogrammation pour gamma 60. Chiffres **2** (1959), 123–135. (English, German and Russian summary)

The author describes an algebraic compiler language of the pre-ALGOL type, like FORTRAN and AT-3, designed for the Γ-60. The key-words are in French, but can be replaced by any other words. The compiler is capable of handling a flexible library whose subroutines may be written in Γ-60, AP-3 itself, or in an intermediate symbolic code. The algebraic symbols themselves may be ambiguous; for example, + may refer to integral arithmetic (for indices), single precision, double precision, complex, or matrix addition as determined by the type statements for the variables (DEFINITION). All sentences must be enumerated, and some may be labelled. The description is too brief to indicate what restrictions exist on subroutines calling subroutines or calling themselves, but does indicate the existence of a call for parallel action, and a call for functions with multiple exits in lieu of switch declarations.

S. Gorn (Philadelphia, Pa.)

3127:

Findler, N. V. Some remarks on the game "dama" which can be played on a digital computer. Comput. J. **3** (1960/61), 40–44.

3128:

Vaida, Dragoş. Calcul des polynômes irréductibles (mod. 2) au moyen de la machine électronique à calculer CIFA-1. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 447–458. (Romanian. Russian and French summaries)

3129:

Hacques, Gérard. Analogie rhéoélectrique en domaine illimité de la fonction associée d'un potentiel harmonique à symétrie axiale. Ann. Assoc. Internat. Calcul. Anal. **2** (1960), 57–62.

The familiar electrolytic tank representation of a Laplacian field (or stretched rubber membrane, conducting (Teledeltos) paper, or resistive network ...) is necessarily finite in size and, in consequence, it is convenient only to represent correctly functions within boundaries which do not extend to infinity. The principle of inversion can be applied to remove this restriction.

The principle is applied to the analogous representation of axially symmetrical harmonic functions, infinite in extent. The object of this article is to describe the method, to prove it and to present applications.

C. Cherry (London)

MECHANICS OF PARTICLES AND SYSTEMS

See also 3309, 3397, 3560.

3130:

Atkin, R. H. ★Classical dynamics. John Wiley & Sons, New York, 1959. ix+273 pp. $5.25.

A characteristic of this clearly written book is its giving a rather condensed text on the general theory and its emphasizing examples, exercises and examination problems. The presentation is short and to the point and the rich content of the book makes it suitable for a thorough course in classical mechanics. A historical introduction and a chapter on vector algebra are followed by many pages (pp. 46–130) on particle motion; we mention here a short account on disturbed planetary motion. Systems of many particles and the motion of a rigid body in the plane are the next chapters. The motion of a system under impulses is treated carefully, up to Bertrand's and Kelvin's theorems. Lagrange's equations, discussion of the inertia tensor, the motion of a rigid body in space (with the Poinsot motion), gyroscopic and non-holonomic problems (spinning top, stability of the steady procession, gyro-compass) and some ten pages on small oscillations are treated in the last part of the book and may give an idea of its scope.

O. Bottema (Delft)

3131:

Corben, H. C.; Stehle, Philip. ★Classical mechanics. 2nd ed. John Wiley & Sons, Inc., New York-London, 1960. xi+389 pp. $12.00.

The first edition of this work, published ten years ago, was reviewed in MR **13**, 593. In this new edition are included applications to the theory of space-charge limited currents, atmospheric drag, the motion of meteoric dust, variational principles in rocket motion and rotating coordinate systems, noncentral forces, the Boltzmann and Navier-Stokes equations, the inverted pendulum, Thomas precession, and the motion of particles in high energy accelerators—a chapter completely re-written to give some account of recent work in this field. The emphasis in the discussion of relativity is on aspects not treated in detail elsewhere.

The book continues to be an outstanding presentation designed to smooth the transition from classical mechanics to quantum and relativistic mechanics.

W. Freiberger (Providence, R.I.)

3132:

Gumerov, Š. A. On a new formulation of the conditions of applicability of the Hamilton-Jacobi method for nonholonomic conservative systems. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1959**, no. 1, 31–44. (Russian. Uzbek summary)

In some respects the classical Hamilton-Jacobi theory of conservative holonomic dynamical systems may be generalised to the case of non-holonomic conservative systems, and conditions for the applicability of such generalisations were found by I. S. Aržanyh [Dokl. Akad. Nauk. SSSR **87** (1952), 15–18; MR **14**, 694], who, however, restricted himself to systems whose kinetic energy may be represented as a quadratic form in the generalised velocities. The present paper extends these methods to the case when the Lagrangian contains also linear terms in the velocities, but it is assumed that the Lagrangian and the linear equations of constraint do not explicitly depend on the time. Furthermore, all calculations—not furnished in the above-mentioned paper—are here presented in detail. A two-dimensional example is treated fully, and the paper concludes with a thorough discussion of the case of one constraint.

H. Rund (Durban)

3133:

Gresky, A. T. A simple unified field equation and hypothesis. J. Franklin Inst. **269** (1960), 105–124.

3134:

de Jonge, A. E. Richard. The correlation of hinged four-bar straight-line motion devices by means of the Roberts theorem and a new proof of the latter. Ann. New York Acad. Sci. **84** (1960), 75–145.

Part I of the paper gives a historical and critical survey of exact and approximate four-bar straight-line mechanisms, the first being related to the Cardanic motion and the second those of Watt, R. Roberts, Chebyshev, Evans and others to which are added some types found by the author. Part II is devoted to Roberts' theorem about the threefold generation of coupler curves. A geometric proof is given (which in the reviewer's opinion is not essentially different from those found elsewhere). Special cases are carefully treated: when one of the hinges is replaced by a sliding pair a modification becomes necessary. It is well known that in the case of the slider crank the three cognate linkages reduce to two, but for its kinematic inversion the author proves that only one is left; Part III deals with the correlation of straight-line mechanisms by means of Roberts' theorem. What is meant by this can best be illustrated by an example: if the original mechanism is a Watt linkage of a certain type, then one cognate may be a Chebyshev linkage and the other an Evans type. By means of tables and figures the correlation question is completely answered. Part IV is a further investigation of the Roberts theorem configuration. If 1, 2, 3, 4 is the original linkage, 1 being the frame, then 2 and 4 intersect in the pole P, and it is well-known that P and the two

similar points of the cognate linkages are on a straight line. The author proves, by means of the Desargues theorem, that the same is true for the three points Q, which are the intersections of 1 and 3 in the respective linkages. In Part V the correlation is studied by means of the theory of curvature. Some appendices are added; one of them deals with a mechanism which traces simultaneously two approximate straight lines at a given angle.

O. Bottema (Delft)

3135:

Gärtner, Robert. Wirkungszusammenhänge mechanischer Glieder. Z. Angew. Math. Mech. **40** (1960), 368–369.

3136:

Novoselov, V. S. Movement of mechanical systems with varying masses. Vestnik Leningrad. Univ. **15** (1960), no. 1, 132–141. (Russian. English summary)

Author's summary: "A method of working out fundamental equations of motion of holonomic as well as unholonomic mechanical systems with variable masses has been presented provided the equations of connections may change together with the masses. The examples concerning the application of the general statements have been given."

3137:

Novoselov, V. S. Sledge of variable mass motion along horizontal plane. Vestnik Leningrad. Univ. **14** (1959), no. 13, 111–120. (Russian. English summary)

3138:

Bauersfeld, W. Die Bewegungsgesetze des Raumkompasses. Ing.-Arch. **27** (1960), 365–371.

Durch geschickte Verwendung von Vektoren, die vom Erdmittelpunkt zum Mittelpunkt der Schwimmkugel eines Kreiselkompasses weisen, läßt sich auf überraschend einfache Weise die Tatsache der Beschleunigungsunabhängigkeit für die Anzeige des Kompasses nachweisen, sofern die bekannte Schuler-Bedingung erfüllt ist. Die Untersuchung von Lot- und Nordweisung ergibt, daß die Anzeige des Gerätes, abgesehen von dem bekannten Fahrtfehler, auch bei beliebigen Bewegungen des Kompaßträgers schwingungsfrei ist. Bei falschen Anfangsbedingungen kann jedoch die Nordanzeige empfindlich gestört werden.

K. Magnus (Stuttgart)

3139:

Manacorda, Tristano. Sul principio dell'effetto giroscopico per i solidi di massa variabile. Ann. Mat. Pura Appl. (4) **48** (1959), 183–191.

Es werden notwendige und hinreichende Bedingungen abgeleitet, bei deren Erfüllung der Drallsatz für einen symmetrischen Kreisel mit variabler Masse und variablen Trägheitsmomenten seine bekannte klassische Form beibehält. Unter sehr allgemeinen Voraussetzungen wird gezeigt, daß nur für einen schon von C. Battaglia angegebenen Fall die klassische Form $(C-A)r_0k=M_e$ gültig ist. M_e ist der Vektor eines körperfesten äußeren Momentes, das in der Äquatorebene des Kreisels liegt, k ist der Einheitsvektor in Richtung der Figurenachse.

K. Magnus (Stuttgart)

3140:

Metelicyn, I. I. Gyroscopic systems with non-ideal joints. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 1, 3–9. (Russian)

Nach einigen allgemeinen Bemerkungen über die Berücksichtigung Coulombscher Reibungskräfte bei der Berechnung mechanischer Systeme mit Hilfe der Lagrangeschen Gleichungen, wird die Auswirkung dieser Reibungskräfte auf das Verhalten von Kreiselsystemen behandelt. Es wird unter anderem auf die bekannte Erscheinung hingewiesen, daß die nur von der Richtung der Geschwindigkeit, aber nicht von ihrer Größe abhängige Reibungskraft um eine Achse durch schwingende Bewegung des Lagers mit einer geeignet gewählten Frequenz in eine quasiviskose, also der Geschwindigkeit proportionale Reibungskraft, verwandelt werden kann.

K. Magnus (Stuttgart)

3141:

Il'linskiĭ, A. Yu. On the equations of the precessional theory of a gyroscope in the form of equations of motion of the pole in the phase plane. Prikl. Mat. Meh. **23** (1959), 801–809 (Russian); translated as J. Appl. Math. Mech. **23**, 1153–1163.

This author gives a rigorous explanation of the way in which the equations of motion of a gyroscope are to be interpreted. He shows how these equations can be made more exact, and points out how to avoid errors in the treatment of gyroscopic problems. He lists conditions under which the equations of the precessional theory of a gyroscope are valid.

H. P. Thielman (Oxnard, Calif.)

3142:

Bogoyavlenskiĭ, A. A. A particular solution of the problem of the motion of a gyroscope on gimbals. Prikl. Mat. Meh. **23** (1959), 958–960 (Russian); translated as J. Appl. Math. Mech. **23**, 1365–1369.

A particular solution of the problem of motion of a heavy gyroscope on gimbals is investigated for the case in which the axis of rotation of the outer ring is horizontal.

H. P. Thielman (Oxnard, Calif.)

3143:

Novoselov, V. S. Motion of stabilized gyroscopic systems on a moving base. Prikl. Mat. Meh. **23** (1959), 964–967 (Russian); translated as J. Appl. Math. Mech. **23**, 1375–1381.

A justification is given for a well-known and frequently used method in the applied theory of gyroscopes.

H. P. Thielman (Oxnard, Calif.)

3144:

Novoselov, V. S. Investigation of stability of vertical position of variable mass gyroscope. Vestnik Leningrad. Univ. **14** (1959), no. 19, 121–129. (Russian. English summary)

Author's summary: "The equation of 'an apex' of a variable mass gyroscope, in which internal motion of particles is examined. The motion of a gyroscope axis about vertical position is studied. At the coincidence of

the centre of gravity with a pendant point the vertical position of the variable mass gyroscope is shown to be stable, at the situation of the centre of gravity below the pendant point—asymptotically stable. The lack of air resistance as well as the internal motion of the particles in the gyroscope are considered in detail. The investigation of the equations of the first approximation is carried out with the help of a special proving theorem about the approximate solution of the system of linear differential equations with variable coefficients."

3145:

Krementulo, V. V. Investigation of the stability of a gyroscope taking into account the dry friction on the axis of the inner Cardan ring (gimbal). Prikl. Mat. Meh. **23** (1959), 968–970 (Russian); translated as J. Appl. Math. Mech. **23**, 1382–1386.

The author applies the direct method of Liapunov, for the investigation of stability, to certain motions of a gyroscope in a Cardan suspension when there is dry friction on the axis of the gimbal.

H. P. Thielman (Oxnard, Calif.)

3146:

Proskuryakov, A. P. Oscillations of a quasilinear non-autonomous system with one degree of freedom near resonance. Prikl. Mat. Meh. **23** (1959), 851–861 (Russian); translated as J. Appl. Math. Mech. **23**, 1218–1232.

The oscillatory system is described by $\ddot{x}+m^2x=f(t)+\mu F(t, x, \dot{x}, \mu)$, where f and F are periodic in t of period 2π. Poincaré's small parameter method leads to a bifurcation equation for the fundamental amplitudes. In this paper the case when the bifurcation equation has multiple roots is discussed.

J. P. LaSalle (Baltimore, Md.)

3147:

Šimanov, S. N. On the vibration theory of quasilinear systems with lag. Prikl. Mat. Meh. **23** (1959), 836–844 (Russian); translated as J. Appl. Math. Mech. **23**, 1198–1208.

A part of the vibration theory for quasilinear systems of ordinary differential equations is generalized for systems with time lag. Except for the work of Minorsky [C.R. Acad. Sci. Paris **226** (1948), 1122–1124; MR **9**, 511], not much has been done with this problem. The system studied is

$$\dot{x} = \sum_{k=1}^{r} A_k x(t-\tau_k)+f(t)+\mu X(t, x(t-\tau_1), \cdots, x(t-\tau_r), \mu),$$

where x, f, X are n-vectors, A_k is an $n\times n$ constant matrix, $f(t)$ is periodic and continuous of period 2π, and X is periodic and continuous in t of period 2π. The function X has in a suitable region continuous first partial derivatives with respect to x and μ and $\tau_1=0<\tau_2<\cdots<\tau_r<2\pi$. The problem is to determine the periodic solutions of period 2π of the systems which become as μ approaches 0 a periodic solution of the system with $\mu=0$. The existence of such periodic solutions is discussed for both the resonant and non-resonant case. Nothing is said about the stability of the periodic solutions. [See also S. N. Šimanov, Dokl. Akad. Nauk SSSR **125** (1959), 1203–1206; MR **21** #5051.]

J. P. LaSalle (Baltimore, Md.)

3148:

Požarickiĭ, G. K. On the stability of permanent rotations of a rigid body with a fixed point under the action of a Newtonian central force field. Prikl. Mat. Meh. **23** (1959), 792–793 (Russian); translated as J. Appl. Math. Mech. **23**, 1134–1137.

Similarly to the Beleckiĭ's problem [Dokl. Akad. Nauk SSSR **113** (1957), 287–290; MR **19**, 1100], in this paper one treats the stability problem under the action of a Newtonian central force field $\mathbf{F}=-gR^{-1}\cdot\mathbf{R}dm$, where R is the distance from the center of forces to the fixed point, assuming that R is much larger as compared to the dimensions of the rigid body. Since the forces are potential and stationary, and the ideal constraints do not contain time explicitly, the system of the equation of motion leads to an energy integral. Also there exists one cyclic integral regarding the kinetic moment. The analysis of the equations shows that the axes of all permanent rotations coincide with the z-axis and in the moving system are distributed over the cone which is analogous to the Staude cone in the problem of the motion of a heavy rigid body with one fixed point in a uniform gravitational field. The Routh criterion is applied to the analysis. By means of the expression for the total energy of the system two equations are derived. The stability conditions are deduced from the expression of the variable potential energy of the system. They, with the equation above, single out the stable permanent axes. The general problem is not investigated, only the particular case when the ellipsoid of inertia is rotational. The same problem in a uniform gravitational field was studied by Rumyancev [Dokl. Akad. Nauk SSSR **116** (1957), 185–188; MR **19**, 1101].

D. P. Rašković (Belgrade)

3149:

Magiros, Dem. G. On a problem of nonlinear mechanics. I. Prakt. Akad. Athēnōn **34** (1959), 238–242. (Greek summary)

This paper analyzes the behavior of a system whose equation of motion is

$$\ddot{Q}+k\dot{Q}+c_1Q+c_2Q^2+c_3Q^3 = B\sin(2t).$$

The coefficients of the non-linear terms are not assumed to be small. Conditions that must be satisfied by the coefficients in order that the system may oscillate with a frequency half of that of the external force are obtained. The amplitudes of the subharmonics and their components and the bounds for the amplitude of the external force are found in terms of the coefficients of the equation. The possibility of having subharmonic motion with different amplitudes is discussed.

L. A. Pipes (Los Angeles, Calif.)

3150:

Nurmia, Matti. A single-station Doppler determination of the orbit of Sputnik I. Arkhimedes **1958**, no. 2, 41–49. (Finnish. English summary)

3151:

Ševelo, V. N. Some remarks on the motion of a variable mass oscillator. Ukrain. Mat. Ž. **11** (1959), 105–108. (Russian)

Es wird die Bewegungsgleichung eines mechanischen

Schwingers mit veränderlicher Masse aufgestellt und für verschiedene einfache Sonderfälle diskutiert. Eine Näherungslösung für den Fall langsamer Massenänderungen wird angegeben. Auf eine technische Anwendung (Schwingungen eines an einem Seil hängenden Gefäßes, dessen Füllung als Funktion der Zeit bekannt ist) wird hingewiesen, jedoch wird keine Lösung angegeben.
K. Magnus (Stuttgart)

3152:

Novoselov, V. S. The fall of a ball drop during evaporation or condensation of vapour on its surface. Vestnik Leningrad. Univ. **15** (1960), no. 7, 116–119. (Russian. English summary)

Author's summary: "Slow as well as quick fall of a ball drop in the immovable gaseous medium in case of evaporation or condensation of vapour on its surface has been considered. In case of slow fall of the drop the law of variable mass in the form of Maxwell is taken, quick fall being backed by the experimental law obtained by Fressling and his followers."

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 3048, 3359, 3527, 3528, 3529, 3530.

3153:

Czerwonko, J. On the generalization of the variational method of Bogoljubov. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 7 (1959), 639–641. (Russian summary, unbound insert)

Ziel der vorliegenden Untersuchung ist die Variationsmethode von Bogoljubov zur angenäherten Berechnung der Zustandssumme (partition function) zu verallgemeinern. Der Verfasser betrachte nach Hölder die $2n$-mal differenzierbare Funktion $\varphi(t)$ und nimmt an, dass $d^{2n}\varphi/dt^{2n} \geqq 0$ ist. $\varphi(t_\nu)$ wird dann in der Umgebung des Punktes $t=M$ in eine Taylorsche Reihe entwickelt und es wird

$$M = (\sum_{\nu=1}^{N} t_\nu p_\nu)(\sum_{\nu=1}^{N} p_\nu)^{-1}$$

gesetzt. Die erwähnte Reihe wird weiter mit p_ν multipliziert und dann nach ν summiert. Setzt man jetzt in der erhaltenen Formel $\varphi(t)=e^{-\beta t}$, $p_\nu=e^{-\beta E_{0\nu}}$, $t_\nu=E_\nu'$ und $E_\nu=E_{0\nu}+E_\nu'$, so erhält man auf der linken Seite die in der statistischen Physik benutzte Zustandssumme (Z) und die rechte Seite, die bis zu dem $(2n-1)$-ten Gliede reicht, ist eine Abschätzung dieser Grösse. Ist $n=1$, so erhält man gerade die Abschätzung von Bogoljubov $Z \geqq Z_0 e^{-\beta M}$, wo $Z_0=\sum_\nu e^{-\beta E_{0\nu}}$ ist. Ausserdem wird auch noch eine obere Grenze für Z hergeleitet.
T. Neugebauer (Budapest)

3154:

Stahl, A. Zur Anwendung des Informationsbegriffes in der statistischen Physik. Z. Naturforschg. **15a** (1960), 655–662.

Author's summary: "Die Arbeit enthält eine Begründung der statistischen Mechanik aus der Informationstheorie. Der Begriff Unordnung wird dabei eliminiert und durch den Begriff der Unkenntnis eines Beobachters ersetzt. Die explizite Berücksichtigung des Beobachters in allen Überlegungen erlaubt einen einfachen Zugang zum Verständnis des Phänomens der Irreversibilität."

3155:

Карлеман, Т. [Carleman, T.] ★Математические задачи кинетической теории газов. [Problèmes mathématiques dans la théorie cinétique des gaz]. Translated from the French by V.-K. I. Karabegov; edited by N. N. Bogolyubov. Biblioteka Sbornika "Matematika". Izdat. Inostr. Lit., Moscow, 1960. 120 pp. 4 r.

Translation of Publ. Sci. Inst. Mittag-Leffler 2 [Almqvist & Wiksells, Uppsala, 1957; MR **20** #4935], with a brief preface by Bogolyubov.

3156:

Amdur, I.; Mason, E. A. Properties of gases at very high temperatures. Phys. Fluids **1** (1958), 370–383.

By employing intermolecular force laws derived from molecular beam scattering experiments, the virial coefficients and transport properties of the rare gases and nitrogen are obtained. The method used offers an approach to the problem of obtaining estimates of gas properties at temperatures too high for direct experiment; the present calculations are valid in the range 1000° to 15,000°K.
S. Simons (London)

3157:

Goldberger, M. L. On the second virial coefficient. Phys. Fluids **2** (1959), 252–255.

Using results from scattering theory, the second virial coefficient is calculated. It is shown that the second virial coefficient is determined by the determinant of the scattering matrix. The method used here is a new one and seems more powerful than the ones normally employed.
D. ter Haar (Oxford)

3158:

Klinger, M. I. The statistical theory of kinetic phenomena. II. Fiz. Tverd. Tela **1** (1959), 1225–1238 (Russian); translated as Soviet Physics. Solid State **1** (1960), 1122–1134.

[For part I see Fiz. Tverd. Tela 1 (1959), 861–872; MR **22** #1143.]

Author's summary: "The general expressions for the kinetic coefficients are obtained by solving the equation of motion of the density matrix of a system subjected to generalized forces, such as temperature gradient T; these general expressions are then employed to obtain approximation formulas for weak-phonon interaction. The formulas thus obtained are used to investigate the thermal conductivity and thermal e.m.f. in semiconductors."
H. Mori (Kyoto)

3159:

Gross, Eugene P.; Jackson, E. Atlee. Kinetic theory of the impulsive motion of an infinite plane. Phys. Fluids **1** (1958), 318–328.

Authors' summary: "The half-range method of solution of the Boltzmann equation is used to give a kinetic theory treatment of the Rayleigh problem. In contrast to the Navier-Stokes or Grad theories, the method yields exact

results for the initial stress and slip velocity at the boundary. It also indicates corrections to the classical Rayleigh results, even at long times compared to the collision periods of the gas molecules. The corrections are shown to be related to the usual slip boundary conditions of hydrodynamics plus an additional part arising from boundary layer effects. The half-range method predicts corrections to the standard methods of between 10%–25% for all values of the time." *H. C. Levey* (Perth)

3160:

Gross, Eugene P.; Jackson, E. Atlee. Kinetic models and the linearized Boltzmann equation. Phys. Fluids **2** (1959), 432–441.

Authors' summary: "Attention is directed to some unsatisfactory features of kinetic theory treatments of problems for which the linearized Boltzmann equation is applicable. The main defects are in the region where nearly free molecular flow conditions prevail. They can be overcome when the problems are treated by simplified kinetic models. In this paper relations between the linearized Boltzmann equation and some models are established. The method is based on a comparison of the eigenvalue spectra of the respective collision operators. Particular attention is paid to inverse fifth molecules. This allows evaluation of the limitations of a given model and shows how more accurate models can be constructed. It is shown how one may overcome the chief shortcomings of approximate solutions of the linearized Boltzmann equation."

N. L. Balazs (Princeton, N.J.)

3161:

Meeron, Emmanuel. Nodal expansions. Distribution functions, potentials of average force, and the Kirkwood superposition approximation. Phys. Fluids **1** (1958), 139–149.

Author's summary: "The customary density power series expansions of potentials of average force and distribution functions are converted into a new class of expansions, defined in terms of topological nodes. The physical meaning of the terms in the new expansions is discussed, and arguments are presented to show that the new expansions can be expected to converge considerably more rapidly than the customary density expansions. The probable occurrence of phase transitions is discussed. Possible further developments are suggested. The Kirkwood superposition approximation is shown to be valid for a large number of terms in the expansions of potentials of average force and distribution functions. The customary comparison of the exact value of the third virial coefficient with that obtained via the Kirkwood superposition approximation is shown to provide no proof of the validity or invalidity of the latter." *H. Mori* (Kyoto)

3162:

Meeron, Emmanuel; Rodemich, Eugene R. Nodal expansions. II. The general expansion of nth order. Phys. Fluids **1** (1958), 246–250.

Authors' summary: "The first-order nodal expansion of potentials of average force and distribution functions, obtained in a previous publication, is summed over again and converted into a second-order expansion. The method is generalized, and recursion formulas are given for nth order expansions of potentials of average force in terms of 'sheaf potentials'. These latter are seen to conform exactly to the Kirkwood superposition principle. The feasibility of actual calculations is briefly discussed. Possible applicability to quantum-mechanical problems (Feynman diagrams) is noted." *H. Mori* (Kyoto)

3163:

Waldmann, L. Diffusionstheorie für polarisierte Teilchen. Z. Naturforschg. **15a** (1960), 19–30.

Author's summary: "The multiple elastic scattering of polarized particles is governed by a generalized Boltzmann equation. The properties of the collision brackets connected with this equation—definiteness and invariances—are studied. Then the state near equilibrium, i.e., nearly isotropic distribution of velocities and spins, is considered (diffusion theory). To describe an ensemble of spin $\frac{1}{2}$-particles one needs in the most simple non-equilibrium approximation two scalars (one pseudo) and four vectors (two pseudo). The scalars are: density of number and helicity. The vectors are: particle stream, density of velocity-spin vector ($\mathbf{v}\times\mathbf{s}$), stream of helicity and density of transverse spin. These scalars and vectors are connected by linear differential equations (diffusion-relaxation-equations). The entropy and the Onsager-Casimir relations are discussed. The theory, supplemented by boundary conditions, is applied on the multiple scattering of spin $\frac{1}{2}$-particles by a thick foil."

3164:

Allnatt, Alan R.; Rice, Stuart A. On the dynamical theory of diffusion in crystals. V. Random-walk treatment of the heat of transport. J. Chem. Phys. **33** (1960), 573–578.

For part IV see Rice and Frisch [same J. **32** (1960), 1026–1034; MR **22** #2198].

Authors' summary: "The heat of transport in a crystalline medium is calculated from the flux of matter in a combined temperature and concentration gradient. The use of a random-walk model leads to the prediction that the heat of transport is exactly equal to the activation energy for motion. This prediction is in excellent agreement with the measurements of Patrick and Lawson of vacancy motion in AgBr. The dynamical interpretation of the theory presented is briefly discussed."

3165:

Prager, Stephen. Diffusion in inhomogeneous media. J. Chem. Phys. **33** (1960), 122–127.

Author's summary: "Methods are given for the calculation of effective diffusion coefficients for inhomogeneous media in which the actual diffusion coefficient varies from point to point in a random manner. An exact result is obtained in the form of an infinite series involving correlations between diffusion coefficients at n different points. A procedure for deriving approximate expressions involving only two-point correlations is also developed and applied to ... diffusion through a porous material."

R. H. Kraichnan (New York)

3166:

Prigogine, I.; Bak, Thor A. Diffusion and chemical reaction in a one-dimensional condensed system. J. Chem. Phys. **31** (1959), 1368–1370.

Authors' summary: "A theory of diffusion processes and chemical reactions based on an integration of the Liouville equation is proposed. A diffusion equation for the probability density in phase space is obtained, and the equation is integrated for a special set of boundary conditions which express that the particle disappears when it reaches a critical energy. It is shown that the rate with which it is annihilated, which is an estimate of the rate of escaping a potential minimum, is characterized by an activation energy, and that the pre-exponential factor is strongly dependent on the frequency for the motion of the particle in the potential minimum. For the special case of an unperturbed potential which is harmonic this can be interpreted as a mass dependence, and it is found that the pre-exponential factor is inversely proportional to m^2."

H. Mori (Kyoto)

3167:

Mason, Edward A.; Vanderslice, Joseph T.; Yos, Jerrold M. Transport properties of high-temperature multicomponent gas mixtures. Phys. Fluids **2** (1959), 688–694.

The kinetic theory of transport in gases is extended to high temperatures, where additional complications exist due to the multiplicity of different interaction energy curves. It is shown that the results of classical kinetic theory can be kept in the same form, but the collision integrals must be computed differently.

S. Simons (London)

3168:

Klimontovič, Yu. L. Relativistic transport equations for a plasma. I. Ž. Èksper. Teoret. Fiz. **37** (1959), 735–744 (Russian); translated as Soviet Physics. JETP **10** (1960), 524–530.

After a distribution function which is a function of the time co-ordinate, the three spatial coordinates, and the four components of the four-momentum is introduced, a transport equation for this function is derived for the case where an electromagnetic field is acting upon the system. A variational principle is given for a relativistic plasma, and the self-consistent-field relativistic equations which lead to relativistically invariant dispersion relations are discussed.

D. ter Haar (Oxford)

3169:

Sodha, Mahendra Singh. Transport phenomena in slightly ionized gases: high electric fields. Phys. Rev. (2) **118** (1960), 378–381.

Author's summary: "Starting with the electron velocity distribution obtained by Chapman and Cowling for a Lorentzian gas, in the presence of an electric field, the author has investigated the variation with electric field of a number of transport properties, arising from a magnetic field, perpendicular to the electric field and temperature gradient in the gas. The applicability of the results to semiconductors has also been pointed out. A constant mean free path has been assumed, which is validated by experiments for helium."

3170:

Yamamoto, Tsunenobu. Quantum statistical mechanical theory of the rate of exchange chemical reactions in the gas phase. J. Chem. Phys. **33** (1960), 281–289.

Author's summary: "A theory of the reaction rate of simple exchange reactions in the gas phase is developed on the basis of the quantum-statistical mechanical theory of linear irreversible processes due to Kubo et al. A formal expression for the rate coefficient is found near the equilibrium point. A number of relations are derived concerning the scattering amplitudes for the collisions involved in the above reactions. By making use of these relations, the rate constant is expressed in terms of the reaction cross sections in a way which coincides with that known from a more intuitive collisional approach. Since no ad hoc assumptions are made, the present theory can be said to provide a statistical mechanical foundation for the collision theory in the particular case discussed."

H. Mori (Kyoto)

3171:

Derr, Vernon E. Irreversibility of systems perturbed by random forces. Phys. Rev. (2) **117** (1960), 1421–1425.

It is shown that macroscopic irreversibility may result from the occurrence of small random perturbing forces, and that such a state of affairs may be described by a "smoothed" density matrix.

D. ter Haar (Oxford)

3172:

Snider, Robert F. Quantum-mechanical modified Boltzmann equation for degenerate internal states. J. Chem. Phys. **32** (1960), 1051–1060.

Author's summary: "A modified quantum-mechanical Boltzmann equation has been derived for the general case in which the molecules have degenerate internal states. This is an equation of the Boltzmann type for a quantity which is simultaneously a Wigner distribution function in molecular phase space, and a density matrix in internal state space. In particular, the nondiagonal terms of this density matrix have been included in the formalism, resulting in the collision term being modified from the usual Boltzmann expression. Thus the collisions are described in terms of combinations of the Lippmann-Schwinger scattering matrix rather than the collision cross section. For nondegenerate states the usual collision term is obtained again."

H. Mori (Kyoto)

3173:

Schram, K.; Nijboer, B. R. A. The Wigner distribution function for systems of bosons or fermions. Physica **25** (1959), 733–741.

Authors' summary: "An expression for the Wigner distribution function valid for systems of bosons or fermions is obtained by making use of correspondence relations between classical quantities and quantum mechanical operators first given by Groenewold. A general and straightforward derivation of the equation of motion for the Wigner distribution function is presented. The equation governing the temperature dependence of the Wigner distribution function in the case of a canonical ensemble can be derived in a completely analogous way."

H. Mori (Kyoto)

3174:

Osipov, A. I. The relaxation of the vibrational motion in an isolated system of harmonic oscillators. Dokl. Akad. Nauk SSSR **130** (1960), 523–525 (Russian); translated as Soviet Physics. Dokl. **5**, 102–104.

3175:

Costa de Beauregard, O. Equivalence entre le principe de l'entropie croissante et le principe des ondes retardées. Rev. Questions Sci. (5) **21** (1960), 41–50.

The paper discusses the equivalence between the law of increasing entropy and the principle of retarded action in classical statistical mechanics and the equivalence between this law and the principle of retarded waves in wave statistical mechanics. *N. Rosen* (Haifa)

3176:

Brown, W. B.; Rowlinson, J. S. A thermodynamic discriminant for the Lennard-Jones potential. Molecular Phys. **3** (1960), 35–47.

It is shown that for molecules interacting with the Lennard-Jones n:6 potential the classical fluctuation discriminant of the configurational energy and the virial can be related to configurational thermodynamic properties and the value of n. The condition that the discriminant is positive provides a lower bound for n. The experimental behavior of argon is examined in the solid, liquid and gas states. It appears that a value of n between 13 and 14 would yield a positive discriminant for the liquid, but that a much higher value is indicated by less reliable results for the gas at high temperature.

A. C. Hurley (Melbourne)

3177a:

Uhlhorn, U. On statistical mechanics of non-equilibrium phenomena. Ark. Fys. **17**, 193–232 (1960).

3177b:

Uhlhorn, Ulf. Macroscopic observables and generalized canonical ensembles. Ark. Fys. **17**, 233–255 (1960).

3177c:

Uhlhorn, U. On the foundations of the linear theory of irreversible processes. I, II. Ark. Fys. **17**, 257–314 (1960).

3177d:

Uhlhorn, Ulf. Statistical mechanical approach to non equilibrium thermodynamics. Ark. Fys. **17**, 343–360 (1960).

This series of papers is based on the construction of a "phase space" that represents the continuous observation of certain macroscopic variables through a chosen period of time. Given the dynamical equations of a mechanical system, in either classical or quantum mechanics, the author constructs a "structure function" [e.g., A. I. Khinchin, *Mathematical foundations of statistical mechanics*, Dover, New York, 1949; MR **10**, 666] and its Laplace-Stieltjes transform, the partition function, and finally its negative logarithm, the Massieu-Planck function. The structure function represents a measure on the above-indicated phase space of the experiment, which is invariant with respect to time displacements and with respect to time reversal.

The following topics are treated after the initial exposition of the author's approach: Extensive and intensive thermodynamic variables; the linear theory of irreversible processes; and reciprocity relations. To this reviewer it appears remarkable that the author has succeeded in deriving many of the well-known relations of irreversible thermodynamics from much less stringent assumptions than are usually adopted in the literature.

P. G. Bergmann (Syracuse, N.Y.)

ELASTICITY, PLASTICITY

See also 3339, 3366, 3369.

3178:

Sneddon, I. N.; Hill, R. (Editors). ★Progress in solid mechanics. Vol. I. North-Holland Publishing Co., Amsterdam; Interscience Publishers Inc., New York; 1960. xii + 448 pp. $15.50.

The eight papers of this first volume of a new series will be reviewed individually.

3179:

Goodier, J. Norman; Hoff, Nicholas J. (Editors). ★Structural mechanics. Proceedings of the First Symposium on Naval Structural Mechanics, held at Stanford University, California, August 11–14, 1958. Pergamon Press, New York-Oxford-London-Paris, 1960. xi + 594 pp. (32 plates) $8.00.

Eighteen papers, of which those of theoretical interest will be reviewed individually.

3180:

Green, A. E.; Rivlin, R. S. The mechanics of non-linear materials with memory. III. Arch. Rational Mech. Anal. **4**, 387–404 (1960).

The authors continue the work on general constitutive equations commenced in earlier papers [same Arch. **1** (1957), 1–21; **3** (1959), 82–90; MR **20** #2130; **21** #2418]. They start with the quite general assumption that these equations take the form of implicit relations between the stress and its time derivatives and the gradients of displacement, velocity, acceleration and higher order time derivatives of these kinematic quantities at a number of instants of time in a given time interval. The correct invariance requirements are then obtained by imposing the condition that the tensors defining the mechanical and kinematic quantities involved must be independent of superimposed rigid body motions, that is, that the constitutive equations are unaltered by motions which differ from the actual motion at any given instant only by rigid body angular velocities, accelerations, etc. The work is then specialised to the case where the stress components are assumed to be functionals of the displacement, velocity and displacement gradients and their higher order time derivatives. These functionals can be approximated by forms involving multiple integrals, and it is shown that under suitable continuity conditions the theory is equivalent to the theory of Rivlin and Ericksen [J. Rational

Mech. Anal. 4 (1955), 323–425; MR 16, 881] in which the stress components are assumed to be continuous functions of the kinematic quantities evaluated at the current time t. The work represents one of the most general analyses of constitutive equations yet published.

J. E. Adkins (Providence, R.I.)

3181:

Pipkin, A. C.; Rivlin, R. S. Electrical conduction in deformed isotropic materials. J. Mathematical Phys. 1 (1960), 127–130.

The results presented in this paper are an application of theorems derived previously by the same authors [Arch. Rational Mech. Anal. 4 (1959), 129–144; MR 22 #1147]. It is shown that if the current **J** at time t in a deformed isotropic material with center of symmetry is a polynomial function of the electric field **e** and the deformation gradients $\partial x_p/\partial X_q$ at time t, then **J** is given by

$$\mathbf{J} = Q_1\mathbf{e}+Q_2\mathbf{g}\cdot\mathbf{e}+Q_3\mathbf{g}^2\cdot\mathbf{e},$$

where the Q's are polynomials in the scalar invariants $g^{-1/2}$, tr $\mathbf{g}$, tr $\mathbf{g}^2$, tr $\mathbf{g}^3$, $\mathbf{e}\cdot\mathbf{e}$, $\mathbf{e}\cdot\mathbf{g}\cdot\mathbf{e}$, and $\mathbf{e}\cdot\mathbf{g}^2\cdot\mathbf{e}$, and $\mathbf{g}$ is the finite strain measure with components $g_{ij}=(\partial x_i/\partial X_k)\times(\partial x_j/\partial X_k)$.

R. A. Toupin (Washington, D.C.)

3182:

Hellman, Olavi. A certain Dirichlet problem in the theory of elasticity. Arkhimedes **1958**, no. 2, 21–28. (Finnish)

3183:

Bondarenko, B. A. On a class of solutions of dynamical equations in the theory of elasticity. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **21** (1957), 41–49. (Russian)

The dynamical equations of elasticity are shown to have a solution of the form $\mathbf{u}=\mathbf{B}-b\,\mathrm{grad}\,(\mathbf{r}\cdot\mathbf{B}+2c_2^{-2}\partial^2\phi)$, where $\partial=\partial/\partial t$, b is defined numerical constant, and c_2 is the velocity of rotational waves in the solid; **B** satisfies a vector wave equation with velocity c_2 and ϕ satisfies a non-homogeneous scalar wave equation with velocity c_1 (the velocity of dilational waves) and "forcing" term $-b(\mathbf{r}\cdot\mathbf{B})$. Similarly we have solutions of the type $\mathbf{u}=\mathbf{A}-a\,\mathrm{curl}\,(\mathbf{r}\times\mathbf{A}-2c_1^{-2}\partial^2\boldsymbol{\psi})$, where **A** satisfies a vector wave equation with velocity c_1, and $\boldsymbol{\psi}$ a non-homogeneous wave equation with velocity c_2 and "forcing" term $-a(\mathbf{r}\times\mathbf{A})$. The author discusses the forms of displacement vector **u** which can be derived from solutions of this type in which **A** and **B** are double integral solutions of the wave equation of Whittaker type [Whittaker and Watson, *A course of modern analysis*, 4th. ed., University Press, Cambridge, England, 1927; p. 397] and ϕ and $\boldsymbol{\psi}$ are retarded potentials. The author merely states the corresponding forms of u: he neither derives expressions for the components of the stress tensor nor considers any particular initial-value or boundary-value problems.

I. N. Sneddon (Glasgow)

3184:

Kaliski, Sylwester. Fundamental solution for elastic and inelastic anisotropic bodies. Arch. Mech. Stos. 11 (1959), 619–647. (Polish and Russian summaries)

A disturbance coming from any center expands in an infinite medium of general anisotropy exactly as in a finite one as long as the boundary of the latter has not been reached yet, say, for a time $t<C$ (=constant). With this restriction one can develop the fundamental solution of the linear hyperbolic system $\sum L_{ik}u_k=-P_i$ of 2nd (or higher) order partial differential equations (equations of motion) into any complete function series satisfying homogeneous boundary conditions. As an example, a parallelepiped and a 3-dimensional Fourier series of u_k are chosen and the impulse of the form $P_i=P_{0i}\delta(t-0)$ is developed in the same way. The determination of the coefficients satisfying the above differential equations is made simultaneously for 4 (2) coefficients and corresponds to the solution of 4 (2) characteristic equations of 3rd (2nd) degree in the 3 (2) -dimensional case. For the plane problem the convergence of the solution is proved and a simple application is given. The case of inelasticity is treated in an analogous way.

The applicability of this witty method is given when it is possible to represent the required solution by relatively few members of the above development with sufficient accuracy. In principle also other developments than Fourier's should be possible but the determination of the coefficients is not worked out so far. In the author's opinion the restriction $t<C$ is acceptable in most practical cases. This does not appear to me to be so evident.

E. Kröner (Stuttgart)

3185:

Stroppe, Heribert. Versetzungen und Eigenspannungen. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg 4 (1960), 105–108. (Russian and English summaries)

The paper offers a nice short review of the linear continuum theory of dislocations and internal stresses. It follows essentially the representation in the reviewer's monograph *Kontinuumstheorie der Versetzungen und Eigenspannungen* [Springer, Berlin, 1958; MR 20 #2117].

E. Kröner (Stuttgart)

3186:

Segawa, Wataru. Measures of finite strain and stress-strain relations. J. Phys. Soc. Japan 15 (1960), 518–522.

The author formulates the equations of nonlinear elasticity theory in terms of one of the less popular material strain measures. There are infinitely many acceptable material strain measures which might be used as a basis for such an investigation. That used by the author is simpler to calculate from given displacement gradients than are most.

J. L. Ericksen (Baltimore, Md.)

3187:

Knowles, James K. Large amplitude oscillations of a tube of incompressible elastic material. Quart. Appl. Math. 18 (1960/61), 71–77.

The author examines axially symmetric oscillations of a long circular tube of incompressible isotropic elastic material, using a general strain energy function. In particular, free oscillations of the tube under no surface pressures and given initial conditions are discussed. The limiting case of a thin tube is then considered and explicit results are given for a material with strain energy of the Mooney-Rivlin type. The author does not mention temperature effects but he appears to assume that temperature changes are negligible.

A. E. Green (Newcastle-upon-Tyne)

3188:

Teodorescu, P. P.; Predeleanu, M. Über das ebene Problem nichthomogener elastischer Körper. Acta Tech. Acad. Sci. Hungar. **27** (1959), 349–369. (English, French and Russian summaries)

Plane problems concerning elastic physically linear and isotropic non-homogeneous bodies are discussed assuming Young's modulus in the form of an exponential function of position and Poisson's ratio to retain a constant value. Existence of a generalized Airy stress function is established. Particular solutions in the form of polynomials and separable solutions are investigated in more detail.

J. Nowinski (Austin, Tex.)

3189:

Uflyand, Ya. S. A mixed problem in the theory of elasticity for a wedge. Izv. Akad. Nauk SSSR. Otd Tehn. Nauk. Meh. Mašinostr. **1959**, no. 2, 156–158. (Russian)

A plane mixed boundary-value problem for a wedge is investigated, where on one edge of the wedge the displacements are prescribed and on the other the external tractions.

In the Papkovitch-Neuber representation of the displacements two functions have been connected by Cauchy-Riemann equations, and the solution was obtained using Mellin integral transforms. An example involving rigid fixing of one edge and a concentrated normal force on the other edge was solved. In a limit case of a half-plane one obtains a solution in a closed form in terms of elementary functions.

J. Nowinski (Austin, Tex.)

3190:

Ganev, Iwan Chr. Vom ebenen Spannungszustand des isotropen elastischen Körpers. Acta Mech. Sinica **4** (1960), 149–168. (Chinese)

A general solution in terms of functions of complex variables is given for plane-stress problems of elasticity involving body forces. (In an appendix the obvious extension to the plane-strain case is mentioned.) Its relation to the well-known results of N. I. Muskhelishvili [*Some basic problems of the mathematical theory of elasticity*, Noordhoff, Groningen, 1953; MR **15**, 370] is not mentioned. Extension of Muskhelishvili's results to the case of non-zero body forces has been carried out by Yi-Yuan Yu [Franklin Inst. **260** (1955), 269–282; MR **17**, 555], and it is believed that the rather complicated results of this paper are reducible to Yu's results which are in a much simpler form. *Yi-Yuan Yu* (Brooklyn, N.Y.)

3191:

Fox, N. Torsion-free stress systems in a thick plate containing a spherical cavity. Quart. J. Mech. Appl. Math. **13** (1960), 228–246.

This paper deals with the second boundary-value problem of classical elastostatics for an infinite plate (of finite thickness), which contains a spherical cavity centered in its middle plane. The problem is treated on the assumption that the loads prescribed on the spherical and plane components of the boundary, as well as the given tractions at infinity, conform to torsion-free axisymmetry about that diameter of the cavity which is perpendicular to the plate faces. The solution is constructed in infinite series form with the aid of Boussinesq's harmonic displacement potentials. The individual terms in this series meet the field equations, obey the requirements at infinity, but lead to progressively diminishing violations of the boundary conditions appropriate to either the spherical or the plane component of the boundary. As specific applications the author considers a plate loaded only at infinity and subjected to (a) isotropic tension or (b) pure bending. In either case numerical results are presented for the peripheral normal stress around the cavity. The results pertaining to case (a) agree with an earlier solution of this particular problem due to Chih-Bing Ling [J. Appl. Mech. **26** (1959), 235–240; MR **21** #3145] of which the present author was apparently unaware.

E. Sternberg (Providence, R.I.)

3192:

Aggarwala, B. D. Elliptic punch on a half space. Z. Angew. Math. Mech. **40** (1960), 374–375.

3193:

Balla, Á. A new solution of the stress conditions in triaxial compression. Acta Tech. Acad. Sci. Hungar. **28** (1960), 349–388. (German, French and Russian summaries)

Author's summary: "The first part of the present paper gives, on the basis of the theory of elasticity, the theoretically correct solution of the stress-conditions of a cylindrical test specimen loaded by triaxial compression.

"The second part presents—in the form of numerical examples—the stresses occurring in the test specimen under the influence of various external loads, discusses the variations in the slenderness of the test specimen and the roughness of the loading plate, and—finally—gives a comparison between its results and the ones of some theoretical researches and experiments."

3194:

An, Bunzai. The elasticity problem of a rigid sphere pressed on the plane surface of a semi-infinite elastic solid. (The effect of friction on the contact surface.) II. Bull. JSME **3** (1960), 47–53.

This is a continuation of a previous paper by the same author [same Bull. **2** (1959), 244–251; MR **21**, 4612]. Formulae for stresses in the interior of the semi-space are derived, numerical and experimental results are given. There are errors in both papers; the expressions for σ_z are different in the two papers, the integrals in the formulae for stresses have wrong values, e.g., according to the author

$$\int_0^\infty (p\cos p-\sin p)p^{-2}J_1(\rho p)dp = \frac{(1-\rho^2)^{1/2}}{2}-\frac{1}{2\rho}\left(\frac{\pi}{2}-\operatorname{arc\,tg}\frac{(1-\rho^2)^{1/2}}{\rho}\right)$$

for $\rho\leq 1$, whereas the true value is $-\frac{1}{4}\pi\rho$. The reviewer feels that a careful verification of the results obtained is necessary. *H. Zorski* (Warsaw)

3195:

Livesley, R. K. The analysis of large structural systems. Comput. J. **3** (1960/61), 34–39.

The paper describes a matrix algorithm for analyzing

structural frames that has been programmed for the EDSAC 2. The mechanical behavior of the members of the frame is supposed to be represented by linear relations between the generalized forces and displacements at the ends of the member. Given the physical constants that define the structure, the computer constructs the matrix of the linear equations on which the analysis of the structure is based. The fact that in the majority of practical frames, joints are only connected to their near neighbors causes the coefficient matrix of these linear equations to contain non-zero elements only in relatively few bands adjoining the principal diagonal. This allows packing of the non-zero elements and thus the handling of large matrices. An efficient packing technique and an elimination method of solving the equations are described. Possible applications to non-linear problems of structural analysis are discussed. *W. Prager* (Providence, R.I.)

3196:

Marguerre, K. Matrices of transmission in beam problems. Progress in solid mechanics, Vol. 1, pp. 59–82. North-Holland Publishing Co., Amsterdam, 1960.

A systematic manner by which the deflection, slope angle, bending moment, and shear force at one point of a beam are related to those at another point by the use of matrix notation is presented. The square matrix of the linear transformation involved has four rows and columns and is termed "matrix of transmission". The use of this matrix greatly facilitates the analysis of n-section beams. The general method is fully described in the paper for the case of the vibrating beam, and the paper concludes with a brief summary of the application of the matrix method to more complex problems. This paper is an excellent introduction to the numerous papers on this subject that have been published in the last few years.

L. A. Pipes (Los Angeles, Calif.)

3197:

Schrader, Alfons. Ein grafisches Verfahren zur Bestimmung des Schubmittelpunktes offener dünnwandiger Profile mit beliebig gekrümmter Mittellinie. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg 4 (1960), 79–85. (Russian and English summaries)

3198:

Borş, C. I. La torsion des barres orthotropes composées et le phénomène de flouage. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. 10 (1959), 157–164. (Romanian. Russian and French summaries)

Author's summary: "Dans cette note, on étudie la torsion des barres cylindriques formées de plusieurs matériaux orthotropes, en tenant compte du phénomène de flouage. (1) La barre est formée de plusieurs tiges cylindriques parallèles, soudées le long des surfaces latérales avec un corps ambiant dans lequel elles se trouvent plantées. La surface latérale extérieure est un cylindre dont les génératrices sont parallèles aux tiges. La section transversale de la barre est formée par les domaines séparés $S_1, S_2, \cdots, S_m$, correspondant aux tiges, et un domaine S_0, correspondant au corps ambiant.

"Le problème conduit à la détermination d'une fonction $F(x, y; t)$, continue dans le domaine $S=S_0+S_1+\cdots+S_m$, y satisfaisant à l'équation

$$\frac{1}{G_{23}^{(i)}(t)}\frac{\partial^2 F}{\partial x^2}+\frac{1}{G_{13}^{(i)}(t)}\frac{\partial^2 F}{\partial y^2}=A_i(t).$$

Sur les courbes de séparation Γ_i, la fonction $F(x, y; t)$ doit satisfaire aux conditions

$$[DF-\int_{\tau_0}^{t} DFw(t,\tau)d\tau]_0=[DF-\int_{\tau_0}^{t} DFw(t,\tau)d\tau]_i$$

sur Γ_i ($i=1, 2, \cdots, m$). (2) On généralise la formule de R. Bredt. (3) On étudie le cas particulier d'une barre à section rectangulaire composée de deux tiges dont les sections, rectangulaires, ont un côté commun."

3199:

Stanišić, M. M.; Hauck, C. A.; Mathias, R. A. Concerning the torsion of prismatic bars having a hexagonal cross-section. J. Aerospace Sci. 27 (1960), 631–633.

3200:

Stanišić, M. M.; Sherwood, B. A. Torsion of an elliptical shaft with diametrically opposite flat sides. J. Aero/Space Sci. 27 (1960), 462–463.

3201:

Ünsaç, Orhan. An investigation for the determination of the cross sectional dimensions of the prismatic bars in the case of uniaxial repeated loading. Bull. Tech. Univ. Istanbul 12 (1960), 85–98. (Turkish summary)

3202:

Bassali, W. A. The torsion of elastic cylinders with regular curvilinear cross sections. J. Math. and Phys. 38 (1959/60), 232–245.

This paper deals with the torsion problem for an elastic isotropic cylinder of uniform simply-connected cross section in the z-plane which is such that it can be mapped on the interior of the unit circle $|\zeta|=1$ in the ζ-plane by the transformation

$$z=c\zeta/(1+m\zeta^n),\quad c>0,$$

where $n\geqq 2$ and m is a real constant such that $-1\leqq m(n-1)\leqq 1$. This restriction on m, n makes the transformation conformal at all points within the boundary.

R. M. Morris (Cardiff)

3203:

Roark, Raymond J. ★Formulas for stress and strain. 3rd ed. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1954. xiii+381 pp. $8.25.

The third edition of this book which has been profitably used by designing engineers has been revised and new material added.

Chapter 1 gives definitions of terms used in elasticity and plasticity. The flexural centre on p. 8 has not been fully defined. The point about which the twist of the section is to vanish is not specified. This has created some confusion in existing literature. The obvious choice is the centroid of the section. A number of other definitions from

continuum mechanics in current use could have been added.

Chapters 3 and 4 give the general principles and methods employed to deal with bodies under stress. No mention is made of finite deformations.

Chapters 7, 8 and 9 deal with formulae concerning beams under simple tension, shear, combined stress, torsion and flexure, chapter 10 with plates, chapter 11 with columns, chapter 12 with pressure vessels and pipes, chapter 13 with bodies under direct bearing and shear stress, chapter 14 with stability and chapter 15 with dynamic and temperature stresses.

A large number of working formulae have been brought together. Both the designers and the theoretical workers will find the new edition good for reference.

B. R. Seth (West Bengal)

3204:

Chang, Chieh C.; Fang, Bertrand T. Initially warped sandwich panel under combined loadings. J. Aerospace Sci. **27** (1960), 779–787.

Authors' summary: "Based on small deflection theory, differential equations for the elastic bending of an orthotropic weak-core sandwich panel with small initial warping are derived by the variational energy method. The applied loads consist of arbitrarily distributed transverse loads and eccentrically applied edge loads and/or edge moments. For the case of a simply supported rectangular panel, solutions of the differential equations are obtained in the form of double Fourier series."

3205:

Szelągowski, Franciszek. A semi-infinite plate acted on by a concentrated moment. Rozprawy Inż. **7** (1959), 551–556. (Polish. Russian and English summaries)

3206:

Scipio, L. Albert, II. Approximate analysis of nonuniform rectangular plates supported at nonequidistant isolated points. Ann. New York Acad. Sci. **79** (1959), 233–256.

In this finite difference solution of the title problem, the plate is divided into panels, the edge moments and shears being obtained from the continuity conditions of deflection and slope. Results agree well with experiment.

H. D. Conway (Ithaca, N.Y.)

3207:

Sen, Bibhutibhusan. Note on the bending of a thin equilateral plate under tension. Z. Angew. Math. Mech. **40** (1960), 276–277.

3208:

Bassali, W. A.; Gorgui, M. A. Flexural problems of circular ring plates and sectorial plates. I. Proc. Cambridge Philos. Soc. **56** (1960), 75–95.

Authors' summary: "In this paper explicit expressions in closed forms are first obtained for the complex potentials and deflexion at any point of a circular annular plate under various edge conditions when the plate is acted upon by general line loadings distributed along the circumference of a concentric circle. These solutions are then used to discuss the bending of a circular plate with a central hole under a concentrated load or a concentrated couple acting at any point of the plate. Solutions for singularly loaded sectorial plates bounded by two arcs of concentric circles and two radii are also derived when the plate is simply supported along the straight edges. The boundary conditions along the circular edges include the cases of a free boundary as well as the elastically restrained boundary which covers the usual rigidly clamped and simply supported boundaries as special cases. The usual restrictions relating to the small deflexion theory of thin plates of constant thickness are assumed. Limiting forms of the resulting solutions are investigated."

R. M. Morris (Cardiff)

3209:

Déév, V. M. On the theory of thick elastic plates. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 252–256. (Ukrainian. Russian and English summaries)

Author's summary: "In this paper the author develops a new method for finding a particular solution of equilibrium equations in displacements permitting the solution of the space problem of elasticity theory for thick plates. Boundary conditions on the lateral surface of the plate are satisfied in the sense of Saint-Venant."

3210:

Nakahara, Ichirō; Koizumi, Takashi. Transverse bending of an infinite plate with a cylindrical circular hole by the three-dimensional theory of elasticity. Bull. JSME **3** (1960), 66–71.

Authors' summary: "In this paper, a three-dimensional solution for an infinite plate with a cylindrical circular hole under transverse bending is given. It is represented by a combination of the solution which is regular in the outside of a hole and the solution which is regular in the infinite plate with finite thickness, and satisfies the boundary conditions on the surface of the hole and the surfaces of the plate. Numerical results are given when the ratios of hole diameter $2a$ to plate thickness $2h$ are about 0.3, 1, and 3. The results show that the smaller a/h becomes, the larger the stress-concentration factor becomes as compared with the value obtained by the theory of thin plates, and the more the circumferential stress across the thickness of the plate at the ends of diameter parallel to the axis of the applied bending moment deviates from a linear distribution."

M. Nassif (Assiut)

3211:

Pîrvu, A. Sur le fléchissement des plaques. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 153–156. (Romanian. Russian and French summaries)

Author's summary: "L'auteur résout le problème du fléchissement élasto-plastique d'une plaque circulaire soumise à un moment uniforme sur le contour."

3212:

Yusuff, S. Design for minimum weight. Considerations based on the long wave instability of stiffened plates in compression. Aircraft Engrg. **32** (1960), 288–294.

Author's summary: "A new approach to the problem of minimum-weight design of stiffened compression panels is

presented. It is predominantly based on the plate instability mode in which the sheet and stiffeners, having been stressed to the same degree, simultaneously buckle over a long wavelength with the length of a buckle equal to the pin-ended length of a panel."

3213:

Bottema, O. Die nichtholonomen Bewegungen einer ebenen Scheibe. Z. Angew. Math. Mech. **40** (1960), 275–276.

3214:

Sen, Bibhutibhusan. Note on the uniqueness of solution of problems connected with thin plates bent by normal pressures. Indian J. Theoret. Phys. **7** (1959), 41–44.

Author's summary: "The object of this note is to prove that the solution of the problem of a thin plate which is in a state of uniform tension, and is bent by normal pressures, is unique for different edge conditions."

3215:

Chen, Shu-tao. Rectangular plates with free edges on elastic foundations. Acta Mech. Sinica **4** (1960), 23–35. (Chinese. English summary)

The classical equations of bending of thin plates on elastic foundation are solved for rectangular plates with free edges, by means of both Ritz's and Galerkin's methods. The plate may be either isotropic or orthotropic, and the loading may be either symmetrical or unsymmetrical. Numerical data are presented so as to facilitate calculations of the deflection and moments.

Yi-Yuan Yu (Brooklyn, N.Y.)

3216:

Saunders, Herbert; Wisniewski, E. J.; Paslay, Paul R. Vibrations of conical shells. J. Acoust. Soc. Amer. **32** (1960), 765–772.

The influence of boundary conditions on the natural frequencies of conical shells of uniform thickness is investigated analytically using a Rayleigh-Ritz procedure. The edge with the smaller radius is considered built in and the other edge either simply supported or free. Boundary conditions were found to have a greater effect on the frequencies of the lower modes of vibration than on those for the higher modes. The results are consistent with previously published experimental and analytical results with which they are compared.

W. D. Kroll (Washington, D.C.)

3217:

Brush, D. O. Strain-energy expressions in nonlinear shell analyses. J. Aero/Space Sci. **27** (1960), 555–556.

It is shown that a variety of strain-energy expressions and strain-displacement relations which have been employed for various thin elastic shell theories can all be obtained by specializing the corresponding general quantities presented by Langhaar [J. Appl. Mech. **16** (1949), 183–189; MR **11**, 288]. *H. B. Keller* (New York)

3218:

Paria, Gunadhar. Thin anisotropic elastic shells. I. Circular cylinder of cylindrical aeolotropy. Bull. Calcutta Math. Soc. **51** (1959), 123–131.

Three differential equations describing the bending of a cylindrically-anisotropic shell are derived. The usual approximations used by Love to the theory of thin shallow shells have been used. The author compares the numerical values of the transversal deflection of a shell made of isotropic and anisotropic materials, respectively, subjected to normal pressure distributions. His statement that the latter are always smaller does not seem to be convincing to the reviewer. *M. Sokolowski* (Warsaw)

3219:

Paria, Gunadhar. Deformation of spherical bodies of fluid-saturated elastic material. I. Bull. Calcutta Math. Soc. **50** (1958), 180–188.

The author notes equations for porous media which are spherically isotropic; that is, which are transversely isotropic with respect to position vectors drawn from some origin. He uses these to analyze deformation of a spherical shell subject to internal and external pressures.

J. L. Ericksen (Baltimore, Md.)

3220:

Paria, Gunadhar. Deformation of porous transversely isotropic elastic material containing a fluid. Bull. Calcutta Math. Soc. **50** (1958), 169–179.

In a series of papers noted by the author, Biot has developed equations to describe the behavior of porous solids. The author specializes these to the case of transversely isotropic media and introduces stress functions which are used to analyze deformation of a half space subject to suddenly applied normal forces.

J. L. Ericksen (Baltimore, Md.)

3221:

Müller, K.-H. Spannungen in anisotropen kreiszylindrischen Rohren. Ing.-Arch. **27** (1960), 417–420.

Using the Muskhelishvili-Lekhnitski complex variable method, stresses in rectilinearly anisotropic circular tube under arbitrary surface tractions are determined assuming plane state of strain. The coefficients in the series representing two fundamental functions of the problem are found, by recurrence, from four linear equations.

J. Nowinski (Austin, Tex.)

3222:

Chakravorti, A. Torsion and bending of an aeolotropic beam having a curtate section. Indian J. Theoret. Phys. **7** (1959), 17–24.

Author's summary: "Following the semi-inverse method of St. Venant components of stress associated with torsion and bending of an aeolotropic beam are obtained in this paper. The cross section of the beam considered is a curvilinear rectangle bounded by two concentric circles and two radii."

3223:

Yacoub, A. R. Elastic equilibrium of completely aeolotropic cylinders; extension by longitudinal lateral loading and end forces. Mathematika **6** (1959), 47–62.

Author's summary: "In the present paper a general solution of the equations of elasticity in complete aeolotropy is found under the assumption that the stresses and therefore the strains are linear in the third cartesian coordinate. This solution is applied to the elastic equilibrium of a completely aeolotropic cylinder, under a distribution of tractions on the lateral surface and resultant forces and couples on the end sections of the cylinder. The problem of extension of a completely aeolotropic cylinder by longitudinal lateral loading and end forces is solved with an application to the elliptic cylinder."
R. M. Morris (Cardiff)

3224:

Dutta, Chinmayee. On the work done during bowing. Indian J. Theoret. Phys. 7 (1959), 5–15.

Author's summary: "The work done by the bow during forward motion of the string and also the work done on the bow during its backward motion are separately calculated for any velocity of bowing. As the works done in the two cases are equal for maintained vibration it is shown that the bowing pressure is constant during forward and backward motion of the string."

3225:

Kovalenko, K. P. The effects of internal and external friction on the dynamic stability of bars. Prikl. Mat. Meh. **23** (1959), 239–248 (Russian); translated as J. Appl. Math. Mech. **23**, 345–358.

The author points out that the equation $d^2y/dt^2+\lambda p(t)y=0$ ($p(t)$ periodic of period π), which comes up in parametric resonance problems where all friction forces are neglected, leads to a physically unsatisfactory description of the phenomena. For instance in the case of bars under action of longitudinally pulsating forces one would conclude that instability arises for arbitrarily small frequencies. To remove this difficulty the author considers the effect of introducing damping forces, and studies the equation $y''+2\mu\nu y'+\mu^2qy=0$, where μ is a positive constant, $\nu>0$ is the (constant) damping factor, q a (say sectionally continuous) non-negative function. The main result states that if q is not identically 0, then for each $\nu>0$, there exists $R(\nu)$ such that for $\mu>R(\nu)$ any solution of the latter differential equation is asymptotically stable.

In the proofs effective use is made of the theory of entire functions. Finally, as an example, the case of stability in prismatic bars under longitudinally pulsating forces is treated. *G. Lumer* (Stanford, Calif.)

3226:

Mitra, A. K. Note on the forced torsional vibration of a cylinder having periodic shearing forces along a ring on the curved surface. Indian J. Theoret. Phys. 7 (1959), 1–4.

Author's summary: "This brief note deals with the investigation of the forced vibration of a cylinder as a result of a shearing stress applied along the circumference of a circle on the surface of the cylinder at a certain height from the lower end of the cylinder which is fixed. Solutions have been obtained (i) when both ends are fixed and (ii) when one end is fixed the other end being free."

3227:

Greenspon, Joshua E. Axially symmetric vibrations of a thick cylindrical shell in an acoustic medium. J. Acoust. Soc. Amer. **32** (1960), 1017–1025.

Author's summary: "This paper treats the dynamic behavior of infinitely long thick cylindrical shells surrounded by water. The shell is excited by axially symmetric forces. The solution is formed by coupling the three-dimensional elastic solution with the expression for the fluid pressure in cylindrical coordinates. Curves are presented which give the forced vibration amplitude and acoustic pressure in the water as a function of frequency for various modes of vibration of the cylinder."

3228:

Gužovs'kiĭ, V. V. On the stability and free oscillations of thin-walled bar systems. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 953–958. (Ukrainian. Russian and English summaries)

Author's summary: "Methods are outlined for determining the critical load and first frequency of free oscillations of thin-walled sets of bars of arbitrary cross section having an open shape.

"Simplicity of solution is attained by considering the deformations of relatively instantaneous coordinates of the centre of rotation.

"It proved possible to determine the transcendental functions from the known tables in the formulae of the method of deformations."

3229:

Andrews, G. J. Vibration isolation of a rigid body on resilient supports. J. Acoust. Soc. Amer. **32** (1960), 995–1001.

Author's summary: "The equations of motion of a rigid body supported by an arbitrary number of arbitrarily oriented and located resilient mounts with damping are developed with rigor. Their solution is given in a form suitable for high-speed electronic digital computer programing and includes prediction of natural frequencies, static displacements and load distribution to the mounts due to static loading, and frequency response curves for a given isolation system. IBM 704 and 709 Fortran computational programs will be supplied on request. An appendix is also included for transforming the inertial properties of a rigid body to a desired frame of reference."

3230:

Więckowski, Józef. The phenomenon of resonance in semi-infinite elastic beams. Rozprawy Inż. **7** (1959), 415–442. (Polish. Russian and English summaries)

An engineering investigation of forced linear vibrations of a semi-infinite elastic beam is carried out; it is assumed that the initial conditions are homogeneous and the inertia of rotation is taken into account. The constraint forces are examined for various types of external kinematical conditions. Certain properties of "resonance", influence of the inertia of rotation and the behaviour of the mechanical energy are deduced. *H. Zorski* (Warsaw)

3231:

Kabulov, V. K. Application of integral equations of balance type to the investigation of oscillation of uncut beams. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1959**, no. 1, 7–15. (Russian. Uzbek summary)

3232:

Bleich, H. H. Dynamic interaction between structures and fluid. Structural mechanics, pp. 263–284. Pergamon Press, New York, 1960.

An expository treatment of two topics: the structural effects of acoustic shock waves and sound radiation resulting from forced vibration of a structure.

R. N. Goss (San Diego, Calif.)

3233:

Ripianu, Andrei. Contribution à l'étude des vibrations transversales des cordes. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 435–446. (Romanian. Russian and French summaries)

Author's summary: "Le problème que l'auteur pose est celui de trouver une méthode de calcul permettant de déterminer la position à chaque instant d'une corde en état de vibration, l'une de ses extrémités étant fixe et l'autre exécutant un mouvement rectiligne suivant une loi quelconque."

3234:

Durant, N. J. Stress in a dynamically loaded helical spring. Quart. J. Mech. Appl. Math. **13** (1960), 251–256.

This purports to be an analysis of the rather elementary case of the axial motion of a helical spring, treated as a one-dimensional homogeneous bar, fixed at one end and subjected to a prescribed harmonic motion at the other. But instead of prescribing this motion as one of the boundary conditions, the author substitutes an equilibrium condition involving the elastic force in the rod at the end, the inertial force on the masses attached to the end (including a third of the mass of the spring, although the use of the wave equation takes account of the spring's inertia) and, instead of the unknown driving force, a force equal to that required to deform the spring statically. If the reviewer has not misread the author this makes no sense.

L. H. Donnell (Ann Arbor, Mich.)

3235:

Flax, A. H. Aero and hydro-elasticity. Structural mechanics, pp. 285–333. Pergamon Press, New York, 1960.

This survey paper deals almost entirely with low-speed aeroelastic problems that may have counterparts in ship (both surface and submarine) design. These problems typically lead to non-self-adjoint equations, and the appropriate extensions of the classical Rayleigh-Ritz and Galerkin methods are described in some detail. Bibliographic coverage is spotty, despite 81 references, but this only indicates the breadth of this essentially post-war field. An extended discussion of possible hydroelastic problems is appended by S. R. Heller and H. N. Abramson.

J. W. Miles (Los Angeles, Calif.)

3236:

Hancock, G. J. Divergence of plate airfoils of low aspect ratio at supersonic speeds. J. Aero/Space Sci. **26** (1959), 495–507, 517.

The state of stress and deformation of low aspect ratio wings of constant thickness and rectangular cross section is analyzed mathematically by the methods of minimum energy. It is assumed in the analysis that the spanwise form of the structural distortion is known and the chordwise distortion is arbitrary. The aerodynamic forces are assumed to be given by the supersonic linearized theory. The problem is also investigated by the use of measured structural flexibility coefficients together with aerodynamic coefficients. The paper is concluded by the use of the minimum energy method and the Rayleigh-Ritz approach.

The analysis indicates that sweeping the leading edge of a plate airfoil of constant thickness increases its stability. For angles of sweep less than 30°, the critical conditions occur when the leading edge is sonic, but for angles greater than 30°, the critical conditions occur when $M=1$.

L. A. Pipes (Los Angeles, Calif.)

3237:

Argyris, J. H.; Kelsey, S. The analysis of fuselages of arbitrary cross-section and taper. Aircraft Engrg. **31** (1959), 169–180, 192–203, 244–256, 272–283.

This paper discusses the analysis of the stresses produced in an aircraft structure by the attachment of loads to the fins of the structure. The effect of frame loads to the overall fuselage loading is discussed in detail. Matrix equations for the calculation of the statically equivalent wing reactions and of the fuselage loading preparatory to the analysis of the statically equivalent wing reactions and of the fuselage moment diagrams are given. The analysis is greatly systematized by the use of matrix algebra and is ideally suited for machine calculations.

L. A. Pipes (Los Angeles, Calif.)

3238:

Barenblatt, G. I. Concerning equilibrium cracks forming during brittle fracture. The stability of isolated cracks. Relationships with energetic theories. Prikl. Mat. Meh. **23** (1959), 893–900 (Russian); translated as J. Appl. Math. Mech. **23**, 1273–1282.

In this paper the theory of brittle cracks introduced by the author [Prikl. Mat. Meh. **23** (1959), 706–721; MR **22** #1168] is compared with the Griffith-Irwin theory.

J. W. Craggs (Newcastle-upon-Tyne)

3239:

Pugsley, Alfred; Macaulay, M. The large-scale crumpling of thin cylindrical columns. Quart. J. Mech. Appl. Math. **13** (1960), 1–9.

When a very thin metal tube of cylindrical section is compressed between parallel plates, its walls tend to buckle in and out to form a diamond pattern of deformation around the tube. The subsequent behaviour is considered when the compression is continued to cause large-scale crumpling of the tube walls. The nature and mode of this crumpling is examined in the light of experimental results and, guided by a simple approximate theory, an

empirical expression for the load to effect the crumpling is obtained.

Reference is made to further work in hand and to work on thicker tubes about to be published elsewhere.

L. S. D. Morley (Farnborough)

3240:

Davids, Norman (Editor). ★International symposium on stress wave propagation in materials. Interscience Publishers, New York-London, 1960. xiii + 337 pp. $10.00.

Fifteen lectures given at Pennsylvania State University, June 30–July 2, 1959, at a Symposium sponsored by the Office of Ordnance Research, U.S. Army. Those of theoretical interest will be reviewed individually.

3241:

Konenekov, Yu. K. A Rayleigh-type flexural wave. Akust. Ž. **6** (1960), 124–126 (Russian); translated as Soviet Physics. Acoust. **6**, 122–125.

3242:

Einspruch, Norman G.; Witterholt, E. J.; Truell, Rohn. Scattering of a plane transverse wave by a spherical obstacle in an elastic medium. J. Appl. Phys. **31** (1960), 806–818.

Here is a complete analysis of the scattering of a plane shear wave propagating in an isotropic elastic medium by an embedded isotropic elastic sphere. Incident and scattered waves are written in terms of spherical harmonics with the proper radial functions, and satisfaction of the displacement and stress continuity conditions leads to the relations between the respective expansion coefficients, in general, in matrix form. The three simpler special cases: rigid, soft (cavity), and liquid spheres are considered especially. The analysis yields the respective scattering cross-sections in terms of sums over the absolute squares of the expansion coefficients. In first (Rayleigh) approximation, the well known fourth-power law is obtained. *H. G. Baerwald* (Albuquerque, N.M.)

3243:

Brekhovskikh, Leonid M. ★Waves in layered media. Translated from the Russian by David Lieberman; translation edited by Robert T. Beyer. Applied Mathematics and Mechanics, Vol. 6. Academic Press, New York-London, 1960. xi + 561 pp. $16.00.

English translation, with the cooperation of the author, of the Russian work [Izdat. Akad. Nauk SSSR, Moscow, 1957] reviewed in MR **20** #1476.

3244:

Gel'činskiĭ, B. Ya. Some questions in the propagation of waves in a homogeneous and isotropic elastic sphere. I. Leningrad. Gos. Univ. Uč. Zap. Ser. Mat. Nauk **32** (1958), 322–345. (Russian)

The author deals with the propagation of the Rayleigh waves across the surface of a sphere, the center of perturbation lying inside the sphere. The solution has the form of double series of Mellin integrals, the final displacements being separated into two parts. To discuss the first part, connected with lower frequencies, the method of residues is used; to the high frequency part of the solution asymptotic methods can be applied. An ample discussion of the behaviour of the surface waves tries to explain some effects observed in seismic investigations.

M. Sokolowski (Warsaw)

3245:

Pavlenko, A. L. On the propagation of discontinuities in a flexible thread. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 4, 112–122. (Russian)

The problem has been solved for two kinds of discontinuities: the weak ones (both displacements and density remain continuous) and strong ones (both coordinates are continuous). All their properties (e.g., the velocity of their propagation) have been found, depending on the kind of discontinuity (weak or strong, longitudinal or transversal). Thermodynamical aspects of the problem have been discussed. *M. Sokolowski* (Warsaw)

3246:

Haĭkovič, I. M.; Halfin, L. A. Effective dynamical parameters of non-homogeneous elastic media in the propagation of a plane longitudinal wave. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1959**, 505–515. (Russian)

The problem of effective dynamical parameters of a non-homogenecus medium in the propagation of seismical signals is studied. To illustrate the non-homogeneity of rocks, the authors consider a theoretical medium composed of two homogeneous elastic bodies: in a homogeneous elastic medium there are spheres of another homogeneous elastic medium, situated in the vertices of a net. For each non-homogeneous medium the effective dynamical parameters depend also on the type of the propagating waves. The authors consider plane, monochromatic longitudinal waves. The conclusion is that the non-homogeneous medium becomes absorbing and dispersing.

N. Cristescu (Bucharest)

3247:

Ang, Dang Dinh. Elastic waves generated by a force moving along a crack. J. Math. and Phys. **38** (1959/60), 246–256.

The author considers elastic waves in the region $-\pi<\theta<\pi$ (cylindrical polar coordinates), due to constant line loads at $r=v_0t$, $\theta=\pm\pi$ where v_0 is constant. The problem is solved by a Laplace transform with respect to time, followed by a Wiener-Hopf technique. The inversion of the Laplace transforms is performed explicitly only for one stress component, and even here the answer is left in a somewhat intractable form.

J. W. Craggs (Newcastle-upon-Tyne)

3248:

Filippov, A. F. A three-dimensional problem of diffraction of an elastic wave at a sharp edge. Prikl. Mat. Meh. **23** (1959), 691–696 (Russian); translated as J. Appl. Math. Mech. **23**, 989–996.

This paper provides an exact solution of the problem of the diffraction of plane longitudinal elastic waves incident on a rigid half-plane. If rectangular cartesian coordinates are chosen so that the half-plane is $y=0$, $x>0$, the displacement vector is of the form grad φ + curl ψ, where $\nabla^2\varphi=a^2\partial^2\varphi/\partial t^2$, $\nabla^2\psi=b^2\partial^2\psi/\partial t^2$ and vanishes on the half-plane.

In the incident waves ψ vanishes and φ is of the form $f(t-cz-c_1x+c_2y)$ where $c(>0)$, c_1, c_2 are constants and $f(s)$ vanishes for $s \leqq 0$. It suffices to consider the case when $f(s)=s$ for $s \geqq 0$.

The solution depends on the solution of the analogous acoustic diffraction problem given by Sobolev in a 1937 Russian translation of volume 2 of Frank and von Mises' (eds.) book, *Die Differential- und Integralgleichungen der Mechanik und Physik* [Glavnaya redakciya obšč. literatury, 1937]; there is no mention of Sobolev in the 1935 German original [Vieweg, Braunschweig, 1935].

E. T. Copson (St. Andrews)

3249:

Tanimoto, B. On the displacement-function for the axially symmetrical visco-elastic vibration. Z. Angew. Math. Mech. **40** (1960), 189–190.

3250:

Lee, E. H. The theory of wave propagation in anelastic materials. International symposium on stress wave propagation in materials, pp. 199–228. Interscience Publishers, New York, 1960.

The analysis of the propagation of one-dimensional stress waves in plastically deforming media and visco-elastic solids is reviewed in conjunction with the available experimental evidence.

A discussion of recent experimental work, in which small amplitude stress waves are propagated into metallic specimens pre-loaded beyond the yield limit, suggests the necessity to include a strain rate sensitivity factor into the incremental flow rule of plasticity. In attempting to verify any proposed version of such a factor, th[illegible] author emphasizes the importance to be placed on obta[illegible] transient strain records, rather than just the final strain distribution at the termination of an impact experiment.

S. C. Hunter (Stanford, Calif.)

3251:

Paria, Gunadhar. Rotatory flow of viscoplastic material of Bingham type. I. Flow between coaxial circular cylinders. Bull. Calcutta Math. Soc. **51** (1959), 116–122.

After briefly surveying work on Bingham materials, the author investigates their behavior in unsteady flow between coaxial cylinders. For the case where the inner cylinder is at rest, the outer is uniformly accelerated and the material is initially at rest, he obtains a formal solution not involving plug flow and indicates when it is meaningful.

J. L. Ericksen (Baltimore, Md.)

3252:

Sokolovsky, Wadim. Écoulement longitudinal d'un milieu plastique entre deux cylindres non circulaires. C. R. Acad. Sci. Paris **249** (1959), 2713–2715.

Flow, described in the title, due to relative motion of two long, rough coaxial cylinders is studied. Stress field in the process of plastic deformation is described by two shear components (normal stresses being equal and constant do not therefore influence the yielding). To solve the problem, two functions of x, y coordinates are introduced. One of them is the deformation function φ, satisfying kinematical conditions of the problem. The other is the stress function ψ, which satisfies equilibrium requirements. For a certain stress-strain velocity relation equations are transformed into a complex plane, and a complex function $\omega=\varphi+i\psi$ is introduced to solve the problem. No particular example is studied. For an extended version of the paper see Prikl. Mat. Meh. **23** (1959), 732–739 [MR **22** #1213] where some examples are given.

A. Sawczuk (Warsaw)

3253:

Komkov, W. On theoretical considerations of plastic-flow criteria. J. Aero/Space Sci. **27** (1960), 477–478.

3254:

Hopkins, H. G. Dynamic expansion of spherical cavities in metals. Progress in solid mechanics, Vol. 1, pp. 83–164. North-Holland Publishing Co., Amsterdam, 1960.

Problems of spherically symmetric dynamic flow in the mathematical theory of plasticity have attracted much attention in recent years. While interest in the subject stems partly from physical problems (principally the formation of internal cavities by the detonation of buried explosive material), the predominant attraction of these problems derives from their simplifying geometrical character, which permits a detailed study of the effects of work-hardening, compressibility and finite deformation on the propagation of stress waves in plastically deforming solids. There are few other cases in the theory of plasticity where, for example, problems of rapid finite deformation of incompressible solids, or the propagation of stress waves in compressible work-hardening materials, prove susceptible to an almost completely analytical treatment.

The present article is a comprehensive account of the propagation of radially divergent disturbances initiated by a transient pressure pulse at the surface of an internal cavity. Particular topics discussed are the propagation of elastic waves, the initiation of dynamic yielding, the propagation of small amplitude waves in elastic-plastic media, the influence of work hardening and the large strain dynamics of incompressible solids. The author concludes with a discussion of the encompassing problem, i.e., the large rapid expansion of a cavity in a work hardening, compressible, elastic-plastic solid; for this it appears that only a numerical solution is possible.

Most of the work reported is recent in origin and has only been previously available in internal Government reports.

The article is particularly valuable in indicating some of the difficulties likely to be encountered in dealing with stress wave problems in situations of more complicated geometry. In this context it is particularly of interest to compare the different characteristics in the behavior of compressible and incompressible solids; while the assumption of incompressibility leads to much welcome mathematical simplification, the results obtained are quite misleading as a guide to the more general case.

An extensive bibliography is appended.

S. C. Hunter (Stanford, Calif.)

3255:

Sawczuk, Antoni; Hodge, Philip G., Jr. Comparison of yield conditions for circular cylindrical shells. J. Franklin Inst. **269** (1960), 362–374.

The problem of a ring of force applied to a long cylindrical shell is considered. Solutions are obtained for uniform and sandwich shells with both the von Mises and Tresca yield conditions. Results are also obtained for a simplified interaction curve and comparison is made between the various yield curves. It is found that the results for the simplified yield curve give a good approximation to those for the von Mises yield curve in this problem.

G. Eason (Newcastle-upon-Tyne)

3256:

Koiter, W. T. General theorems for elastic-plastic solids. Progress in solid mechanics, Vol. 1, pp. 165–221. North-Holland Publishing Co., Amsterdam, 1960.

This 50-page article is divided into seven chapters and contains over 80 references to other works. It is entirely concerned with the theory of small elastic-plastic deformations and does not contain any structural examples. Some of the theory presented represents original contributions of the author.

The table of contents gives an accurate and concise picture of the material covered: Introduction; Basic assumptions and stress-strain relations; Uniqueness theorems; Minimum principles; Plastic collapse theorems and limit analysis; Shakedown theorems; and Existence of solutions. Both perfectly plastic and work-hardening materials are considered. Drucker's quasi-thermodynamic principle is taken as a starting point and all results are thus presented from a unified point of view. Most sections conclude with a historical section in which credit is given for the various individual contributions to the resultant theory.

P. G. Hodge, Jr. (Chicago, Ill.)

3257:

Strunin, B. M. The inhomogeneity of plastic deformation in extension. Dokl. Akad. Nauk SSSR **130** (1960), 310–313 (Russian); translated as Soviet Physics. Dokl. **5**, 151–154.

3258:

Prager, William. Stress analysis in the plastic range. Frontiers of numerical mathematics, pp. 3–21. University of Wisconsin Press, Madison, Wis., 1960.

This paper is a review of the theory of plasticity with particular reference to a simple truss problem which illustrates most of the concepts found in a more complicated fashion in more general problems. The truss considered has an arbitrary number of bars but is so constructed that there is precisely one redundant bar. An ingenious geometrical representation is given in a two-dimensional space whose coordinates are essentially the load magnitude and a measure of the deviation from a purely elastic response. The response of the truss to a (slowly) time-varying load can then be pictured by the motion of a "stress-point" in this space, subject to geometrically expressed equilibrium, yield, and stress-strain relations.

Illustrations are given for a perfectly plastic and a work-hardening truss. In the former case it is shown that the general problem of stress determination can be expressed as a problem in "quadratic programming" but that the determination of the load carrying capacity of the truss reduces to a problem in "linear programming". Also discussed are the problems of shakedown and limit design. The determination of stresses in the work-hardening truss can be reduced to one in "differential quadratic programming".

The paper was presented at a meeting on numerical analysis at the University of Wisconsin. The written version includes a discussion of the paper by the author and several members of the audience. Most of the discussion is concerned with the problems involved in generalizing the simple problem presented to more complex situations.

P. G. Hodge, Jr. (Chicago, Ill.)

3259:

Kačanov, L. M. Variational methods of solution of plasticity problems. Prikl. Mat. Meh. **23** (1959), 616–617 (Russian); translated as J. Appl. Math. Mech. **23**, 880–883.

'Plasticity' in the title is misleading. The author actually considers only a very special non-linear elastic stress-strain law, in which the stress and strain deviators are proportional, with a modulus depending on the second stress invariant. For this particular solid (and indeed for more general ones) well-known variation principles exist when the strain is infinitesimal. The author briefly sketches, without examples, an iterative method by which these might be used to furnish approximations to a general boundary-value problem.

R. Hill (Nottingham)

3260:

Shiel[illegible] R. T. Plate design for minimum weight. Q[illegible]. Appl. Math. **18** (1960/61), 131–144.

It is required to design a plate, using minimum volume of material, to sustain a given transverse distribution of pressure, when the shape is prescribed together with the support conditions along the edge. Here the plate is a sandwich, consisting of a core of given thickness H (carrying shear force only) and identical face sheets of variable thickness $h \ll H$ (carrying direct stress only). The face material is rigid/plastic with Tresca yield and potential functions. Basic equations are formulated for any shape of plate, and four different types of solution are detailed. An elliptical plate is taken as a worked example.

R. Hill (Nottingham)

3261:

Shield, R. T. On the optimum design of shells. J. Appl. Mech. **27** (1960), 316–322.

A procedure is developed for obtaining the optimum design of an elastic, perfectly plastic shell or structure which supports prescribed loads and which is the optimum design for a given criterion. Using theorems of limit-analysis, bounds on the load-carrying capacity of a given shell for a given type of loading are obtained. The action of body forces is included and no restriction of homogeneity of material is imposed. For illustration, some problems for the minimum volume design of a circular cylindrical sandwich shell are solved. It is found that only for relatively short shells does the minimum volume design effect an appreciable saving over the membrane design.

N. Cristescu (Bucharest)

3262:

Rozenblyum, V. I. On approximate equations of creep. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. 1959, no. 5, 157–160. (Russian)

Application is made of certain approximate equations for steady incompressible creep, based upon Tresca's yield criterion and associated flow rule, to the analysis of simple problems for a plate with a circular hole, under conditions of circular symmetry. Particular attention is given to the variation of results with the exponent chosen in the creep relation. *H. G. Hopkins* (Sevenoaks)

3263:

Taubert, Paul. Eine Kriechtheorie für Metalle unter Berücksichtigung von Verfestigung und Erholung. Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Phys. Tech. **1958**, no. 7, 58 pp. (1959).

The author develops a new theory of creep of metals in which strain hardening and recovery are taken into account. It is an interesting generalization of the exhaustion theory of creep. The author gives a very complete theoretical description of the creep phenomenon.

A. Phillips (New Haven, Conn.)

3264:

Olszak, Waclaw. On some basic aspects of the theory of non-homogeneous loose and cohesive media. Arch. Mech. Stos. **11** (1959), 751–766. (Polish and Russian summaries)

This paper is concerned with continuous, perfectly plastic media which obey a yield condition of the form

$$M_1(P)J_1+M_2(P)J_2^{n/2} = M_0(P),$$

where M_0, M_1, M_2 are functions of the position of a current point P in the material, J_1, J_2 are the first and second order invariants of the stress tensor, and n is an integer. The medium is thus assumed to be isotropic but not, in general, homogeneous. This yield condition represents, in principal stress space, an nth order paraboloid of revolution. The author considers in some detail the interpretation of the cases $n=2$, $n=1$ in which the yield surface reduces respectively to an ordinary paraboloid of revolution and a circular cone, and shows, in general terms, how results previously obtained for homogeneous media are affected by inhomogeneity.

The reviewer feels that the omission of the third order stress invariant J_3 places a severe limitation on this work. The author erroneously states that the case in which the yield surface is conical corresponds to Coulomb's yield condition for soils. In fact, as shown by R. T. Shield [J. Mech. Phys. Solids **4** (1955), 10–16; MR **17**, 805], the Coulomb condition depends upon J_3, and the associated yield surface is a hexagonal pyramid.

P. Chadwick (Sheffield)

3265:

Boley, Bruno A.; Weiner, Jerome H. ★Theory of thermal stresses. John Wiley & Sons, Inc., New York-London, 1960. xvi+586 pp. $15.50.

With the increasing importance of thermal stress problems in engineering science the need for a summary of the present state of our knowledge in the field has been growing rapidly. The only text available in English, the book by Gatewood [*Thermal stresses*, McGraw-Hill, New York, 1957; MR **19**, 1001], is perhaps too restricted in its scope to fill this gap. In contrast the book under review is much more general in the presentation of the extensive material to which both authors have made outstanding contributions. Written in a rather broad style, it is divided in two main parts. The first part obviously aims at the scientist and the researcher. Here the mechanical and thermodynamical foundations of the theory are laid. Unfortunately the value of these general considerations is considerably reduced by the fact that they are restricted from the outset to small deformations and to linear stress-strain laws.

A number of uniqueness proofs are presented in detail. The influence of inertia effects and of the coupling between the heat equation and the stress strain equations is investigated and thoroughly discussed. Illustrative examples are given.

Considerable space is devoted to a presentation of the theory of heat conduction. In view of the existence of such excellent texts on the subject as the book by Carslaw and Jaeger [*Conduction of heat in solids*, Clarendon, Oxford, 1947; MR **9**, 188], some doubts may perhaps be raised as to the necessity of that inclusion.

The second part of the book is almost completely self-contained and can be read without referring to the first. It addresses itself to the practicing engineer and provides a wealth of readily usable material in the form of tables, graphs, formulas and references. It is concerned not only with purely elastic systems but also presents an excellent introduction to linear and nonlinear viscoelastic and to plastic stress analysis. A chapter on thermoelastic stability is included.

The book is a must for everyone working in the field of thermal stress. *H. Parkus* (Vienna)

3266:

Jaunzemis, W.; Sternberg, E. Transient thermal stresses in a semi-infinite slab. J. Appl. Mech. **27** (1960), 93–103.

A segment of the edge of a semi-infinite slab with faces insulated against heat loss is exposed to a sudden uniform change in temperature. The authors first derive solution of the heat equation both in integral form and in infinite-series form. Then they find Airy's stress function by superposing solutions corresponding to suitably distributed centers of dilatation. Stresses are calculated and the formal solution which appears in the form of infinite series is rigorously established. An alternative form more rapidly convergent for large values of time is also presented.

In addition—similar to a previously published paper by the senior author—the influence of the rate of heating is studied by replacing the temperature step-function by a ramp-type function.

The paper is very carefully written and contains extensive numerical results. *H. Parkus* (Vienna)

3267:

Lockett, F. J.; Sneddon, I. N. Propagation of thermal stresses in an infinite medium. Proc. Edinburgh Math. Soc. **11** (1958/59), 237–244.

The general problem of thermoelasticity in the case of

coupling between the mechanical and thermal fields is discussed. Body forces are taken into consideration. The solutions are given in the form of four- or three-dimensional integrals of the Fourier and Hankel type. Many particular cases, e.g., plane and axi-symmetrical thermoelastic fields, quasi-static and uncoupled problems are derived from the general solutions. The equivalence of some body forces and certain heat-source distributions in the quasi-static case is shown. *M. Sokolowski* (Warsaw)

3268:

Sneddon, Ian N. Boundary value problems in thermoelasticity. Boundary problems in differential equations, pp. 231–241. Univ. of Wisconsin Press, Madison, 1960.

This is essentially a review of papers on steady-state thermal stress in the semi-infinite body and the thick plate of infinite radius published elsewhere by the author and his associates. *H. Parkus* (Vienna)

3269:

Nowacki, Witold; Sokołowski, Marek. Propagation of thermoelastic waves in plates. Arch. Mech. Stos. **11** (1959), 715–727. (Polish and Russian summaries)

For an infinite plate a solution of the linearized thermoelastic equations of motion is given representing an elastic wave travelling in one direction parallel to the surfaces of the plate. Coupling between thermoelastic equations and equation of heat conduction is taken into consideration. Surfaces of the plate are assumed free of stress and are either kept at constant temperature or are perfectly insulated against heat loss.

Symmetric and skewsymmetric (bending) motion is discussed. In evaluating the characteristic determinants certain approximations are made. Numerical results for an aluminium plate are presented. *H. Parkus* (Vienna)

3270:

Liu, Hsien-chih. Bestimmung der Wärmespannungen im Halbraum infolge einer punktförmig gedachten Wärmequelle an der Oberfläche mit einem unmittelbaren Rechenverfahren. Acta Mech. Sinica **4** (1960), 80–83.

The well-known solution of the title problem [cf. E. Melan and H. Parkus, *Wärmespannungen infolge stationärer Temperaturfelder*, Springer, Wien, 1953; MR **16**, 306; p. 74] is obtained again by the author by means of what he terms "direct method" avoiding the use of Love's displacement function. *H. Parkus* (Vienna)

3271:

Kneschke, A. Elastische Kreisplatten unter einseitiger Temperatureinwirkung. Z. Angew. Math. Mech. **40** (1960), 40–46.

Der Verfasser gibt die strenge Lösung des thermoelastischen Problems der dicken, durch eine drehsymmetrische stationäre Temperaturverteilung beanspruchten Kreisplatte. Der Oberseite der Platte wird eine vorgegebene, nur vom Radius abhängige Temperatur aufgeprägt, während an der unteren Seite Wärmeübergang in ein Medium konstanter Temperatur stattfindet. Die Mantelfläche der Platte ist als vollkommen wärmeundurchlässig angenommen.

Die beiden Sonderfälle der völlig freien und der am Umfang parallel und reibungsfrei geführten Platte werden näher untersucht. Zahlenergebnisse für eine aus optischem Glas gefertigte Platte (Astrospiegel) werden vorgelegt. *H. Parkus* (Vienna)

3272:

Johns, D. J. Thermal stresses in a long circular shell with axial temperature variation. J. Aero/Space Sci. **27** (1960), 393–394.

In continuation of a number of papers on the problem of thermal stresses in stiffened cylindrical shells the author applies some well-known formulas to the combination cylindrical shell-plane circular bulkhead. *H. Parkus* (Vienna)

3273:

Forray, Marvin; Newman, Malcolm. Axisymmetric bending stresses in solid circular plates with thermal gradients. J. Aerospace Sci. **27** (1960), 717–718.

3274:

Newman, Malcolm; Forray, Marvin. Bending stresses due to temperature in hollow circular plates. I. J. Aerospace Sci. **27** (1960), 792–793.

STRUCTURE OF MATTER

3275:

Penrose, O. A variational method for the ground state of a Bose fluid. Proc. Roy. Soc. London. Ser. A **256** (1960), 106–114.

A special variational principle is used for the ground state of a Bose fluid, based on the circumstance that the wave function can be considered to be real and one-signed. The Schrödinger energy integral is treated as a bilinear functional in the wave function, and is transformed so that the wave function and the particle density can be used as independent variants. The stationary character of the energy functional then serves to connect these quantities as well as is possible.

This procedure leads to a formula for the energy of the ground state in the case of weak interactions between the particles, with relatively simple trial wave functions. Strong interactions can be handled only by numerical calculations which are to be described in another paper. *E. L. Hill* (Minneapolis, Minn.)

3276:

Vénéroni, Marcel; Arvieu, Robert. Quasi-particules et états collectifs des noyaux sphériques. C.R. Acad. Sci. Paris **250** (1960), 2155–2157.

The authors follow up an earlier note [Arvieu and Vénéroni, same C.R. **250** (1960), 992–994; MR **22** #1397] on the application of Bogolyubov's method of canonical transformations (quasi-particles) to the study of collective modes of oscillation of nuclei. Certain operators associated with the quasi-particles are subjected to the commutation rules for bosons. Perturbation methods are used for a general discussion of the structure of the energy level system. *E. L. Hill* (Minneapolis, Minn.)

3277:

Konstantinov, O. V.; Perel', V. I. Quantum theory of spatial dispersion of electric and magnetic susceptibilities. Ž. Èksper. Teoret. Fiz. **37** (1959), 786–792 (Russian); translated as Soviet Physics. JETP **10** (1960), 560–564.

A formal theory of the electromagnetic susceptibility tensors of material media is developed in which the medium is not necessarily homogeneous or isotropic. The electromagnetic field is treated classically, while the matter is described quantum-mechanically by means of the density matrix. The construction of the current density functions is described in some detail.

E. L. Hill (Minneapolis, Minn.)

3278:

Fumi, F. G.; Tosi, M. P. Extension of the Madelung method for the evaluation of lattice sums. Phys. Rev. (2) **117** (1960), 1466–1468.

Die bekannte Madelungsche Methode zur Berechnung der elektrostatischen Energie von Ionengittern wird auf den Fall erweitert, dass die Wechselwirkung von zwei Körpern durch eine Funktion R^{-n} ($n>0$, z.B. van der Waalssche Kräfte) beschrieben wird. Zuerst wird ein lineares Bravaissches Gitter betrachtet. Die Wechselwirkung dieses Gitters mit einem ausserhalb in einer Entfernung r gelegenen Punkt erhält man dann in den gewohnten Bezeichnungen aus der Formel

$$(1)\qquad S_n(x, r) = \sum_{l=-\infty}^{+\infty} \frac{1}{[(x-x_l)^2+r^2]^{n/2}},$$

wo $x_l = la$ und l eine ganze Zahl ist. (1) wird weiter in die Fouriersche Reihe

$$(2)\qquad S_n(x, r) = \frac{1}{a}\sum_{\kappa=-\infty}^{+\infty} e^{i\kappa x}\int_{-\infty}^{+\infty} \frac{e^{-i\kappa x}}{(x^2+r^2)^{n/2}}\,dx$$

entwickelt. (2) kann dann für $\kappa=(2\pi/a)k=0$ mit Hilfe von Γ-Funktionen und für $|k|>0$ ebenfalls durch Γ- und modifizierte Besselfunktionen zweiter Art ausgedrückt werden. Für $n\leqq 1$ divergiert $S_n(x, r)$; lässt man jedoch das Glied $k=0$ weg, so hört diese Schwierigkeit auf und eben das Weglassen dieses Gliedes entspricht dem Fall des mit einer gleichmässigen Ladungsverteilung-neutralisierten Gitters. Analoge Resultate folgen auch für ein zweidimensionales Gitter. Aus diesen Ergebnissen kann man weiter alle in einem dreidimensionalen Gitter auftretenden Probleme berechnen. Einige Anwendungen und schon früher berechnete Fälle werden besprochen. (Bei van der Waalsschen und ähnlichen Wechselwirkungen nehmen die Energieanteile mit der Entfernung selbstverständlich so schnell ab, dass in physikalischen Problemen meistens eine elementare unmittelbare Berechnung zum Ziele führt.)

T. Neugebauer (Budapest)

3279:

Laval, J. Les tensions thermiques dans le milieu cristallin. J. Phys. Radium **20** (1959), 577–588. (English summary)

Author's summary: "Interatomic forces developed by thermal agitation do not obey Hooke's law. Consequently, by oscillating, the atoms exert on one another repulsive forces which are sensibly constant over a period of time. Thermal strains which two parts of a crystal exert on each other are equal to the resultants, referred to unit area, of the repulsive forces exerted by the atoms of one part upon the atoms of the other part. By this definition, it is possible to estimate the thermal tensions as a function of the temperature and of the fundamental properties of the crystalline medium: its potential energy and its elementary translations. We find that, at low and intermediate temperatures, thermal tensions are proportional to the squares of the average quadratic amplitudes of the fundamental thermal oscillations."

3280:

Cekvava, B. E. Generalized Fresnel equations for the surface of a crystal, taking account of anisotropy in the effective exciton mass. Fiz. Tverd. Tela **2** (1960), 482–488 (Russian); translated as Soviet Physics. Solid State **2**, 447–453.

Author's summary: "The Fresnel equations for the transmission of light through a vacuum-crystal boundary are generalized. Equations are obtained for the reflectivity of light from the surface of the crystal, as well as expressions for the coefficients of transparency, reflection, and attenuation of the light intensity in passing through a plane-parallel plate. Experiments are discussed for the purpose of verifying the optical anisotropy of cubic crystals that is obtained in the present paper."

3281:

Seitz, Frederick; Turnbull, David (Editors). ★Solid state physics: Advances in research and applications, Vol. 10. Academic Press, New York-London, 1960. xv+516 pp. $14.50.

A further five articles are added to this invaluable collection. In the first, P. R. Wallace discusses "Position annihilation in solids and liquids". Some theoretical understanding of the influence of the medium has been achieved but there are still many difficulties in the use of position annihilation observations as a tool for investigating solid state problems. Experimental results on a great range of materials are given. In "Diffusion in metals", D. Lazarus concentrates on the basic theory of diffusion and its temperature dependence and emphasizes the basic assumptions that have been made. A brief summary of experimental methods and results follows.

The remaining three papers are also mainly theoretical. B. S. Gourary and F. J. Adrian deal with "Wave functions for electron-excess color centers in alkali halide crystals", limiting themselves to those with the NaCl structure. After a brief summary of the experimental situation, they discuss critically the various approximations that may be applied to the color-center problem and describe in detail the various approaches based on continuum and semi-continuum models, on molecular orbital techniques and on the point-ion-lattice approximation. R. de Wit's article on "The continuum theory of stationary dislocations" concentrates on the powerful methods introduced by E. Kröner, in which field equations are formulated in analogy to Maxwell's equations in electromagnetic theory for dealing with internal stress problems; these are here applied to dislocations and expressions are derived for the energies of dislocations of some simple configurations. The earlier part of the article states the basic elasticity theory and gives a careful definition of a dislocation and its characteristic quantities.

The last three-fifths of the book is taken up by a very extensive article on "Theoretical aspects of superconductivity" by the late M. R. Shafroth. This is divided into two parts, "Phenomenology" and "Microscopic theory", and gives a very thorough account of the present state of the theory of superconductivity. In the former part, more general points of view are considered in addition to the London theory. In the latter part, the earlier theories based on single electrons are reviewed briefly before proceeding to the discussion of pair correlations and the recent theories of Bardeen, Cooper and Shrieffer and of Bogoljubov based on such correlations.

M. S. Paterson (Berkeley, Calif.)

3282:

Low, William. ★Paramagnetic resonance in solids. Solid state physics (edited by Frederick Seitz and David Turnbull), Supplement 2. Academic Press, New York-London, 1960. viii + 212 pp. $7.50.

Paramagnetic resonance spectroscopy is a recently-developed tool which has found much application in solid state physics, both for obtaining energy levels of paramagnetic ions in crystals and for investigating the effect of the local crystalline field on these energy levels. This book gives a valuable discussion of the theory underlying the use of paramagnetic resonance in solving solid state problems which will be useful to workers wishing to apply it. Firstly the theory of the effect of the strong local crystalline electric fields on the energy levels of paramagnetic entities, and the associated symmetry considerations, is reviewed. This is followed by sections on the spectra of the transition elements in single crystals, on the influence giving rise to line broadening, and on the paramagnetic resonance of color centers and other defects in crystals. Finally some principles of the design of spectrometers are given briefly.

M. S. Paterson (Berkeley, Calif.)

3283:

Miąsek, Maria. Electron impact ionization. The probability of occupying states after ionization. Czechoslovak J. Phys. **10** (1960), 584–594. (Russian summary)

Author's summary: "The probability function of occupying states with a definite energy in the conduction band after collision in the process of electron impact ionization is calculated. The valence band is assumed of finite width. Several cases of the primary energy and the effective mass in the valence band are examined."

3284:

Bezirganyan, P. A. Dynamic theory of interference of X-rays in finite crystals. Akad. Nauk Armyan. SSR. Dokl. **29** (1959), 223–230. (Russian. Armenian summary)

3285:

Hauptman, H.; Karle, J. A unified program for phase determination, type $3P_3$. Acta Cryst. **13** (1960), 745–748.

Authors' summary: "The unified program for phase determination, valid for all the space groups and both the equal and unequal atom cases is continued here. The present paper is concerned with the centrosymmetric space groups comprising type $3P_3$. A detailed procedure for phase determination is described for this type."

FLUID MECHANICS, ACOUSTICS

See also A2869, A2870, 3097, 3112, 3113, 3159, 3165, 3227, 3236, 3251, 3397, 3400, 3401, 3340, 3552.

3286:

Milne-Thomson, L. M. ★Theoretical hydrodynamics. 4th ed. The Macmillan Co., New York, 1960. xxviii + 660 pp. $11.00.

This is the fourth edition of the work, whose second edition was published by Milne-Thomson in 1950 [MR **11**, 471], and differs from that in a considerable re-arrangement of the subject matter and in various forms of presentation. Moreover, there are a number of important additions, particularly the application of the Plemelj formulae to certain boundary layer problems and a systematic discussion of flow under gravity with a free surface. [A third edition appeared in 1956 and was reviewed in MR **17**, 796.]

G. Temple (Oxford)

3287:

Li, Ta. Über die partiellen Ableitungen der von Kármán-Tsienschen Potentiale. Z. Angew. Math. Mech. **40** (1960), 370–372.

3288:

Vallander, S. V. Developing wings. Vestnik Leningrad. Univ. **14** (1959), no. 19, 113–120. (Russian. English summary)

Author's summary: "A generalization of plane Prandl-Meyer flows for three-dimensional flows is given. Generalized Prandl-Meyer flows can be applied for the approximate calculations of flows past some wings of finite span. It is necessary to integrate the system of two ordinary differential equations of the first order for the approximate solving of the flow-problem of the wings."

3289:

Vallander, S. V. Numerical calculation of flows past some wings of finite span. Vestnik Leningrad. Univ. **14** (1959), no. 19, 106–112. (Russian. English summary)

Author's summary: "The article deals with the equations and boundary conditions required for the calculation of the flow past some wings of finite span. Nonlinear differential equations of the problem are the equations for functions of two independent variables. In the hyperbolic case one can use the method of characteristics for the numerical calculation of flows past these wings. The assumptions of the paper are quite usual in considering the problems of compressible flows. Interference between wing and fuselage is neglected."

3290:

Bryson, A. E. Symmetric vortex separation on circular cylinders and cones. J. Appl. Mech. **26** (1959), 643–648.

The symmetric vortex separation observed behind a cylinder moving at sufficiently large Reynolds number is analyzed by assuming that each vortex sheet can be approximated by a single concentrated line vortex at the center of gravity of vorticity with a connecting sheet of vanishingly small vorticity to a "feeding point" on the body. A complicated first-order non-linear differential equation for the coordinates of the vortex is integrated numerically. It is shown that the feeding points are unstable equilibrium points in that any small motion will cause the vortex to move away from this point. The cases of vortex separation on an inclined cylinder and on a slender cone are also discussed.

R. C. Di Prima (Troy, N.Y.)

3291:

Iwasaki, Matsunosuke. Vortex theory of an airscrew in consideration of contraction or expansion of slipstream and variation of pitch of vortex sheets in it. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **7** (1959), 159–230.

3292:

Ordway, Donald Earl; Hale, Richard W. Theory of supersonic-propeller aerodynamics. J. Aero/Space Sci. **27** (1960), 437–450.

Authors' summary: "A supersonic propeller with blades attached to an infinite cylinder as a hub is studied. The forward speed may be subsonic, but the relative speed at each section is supersonic. The lightly loaded blades are represented by a surface distribution of appropriate 'modified' sources in a fashion similar to ordinary supersonic thin-wing theory. These sources are found by approximating the exact potential for a constant-strength compressible source traveling along a helical path. The usual relationship between the source strength and boundary condition is found; and subsequently the source distribution is given, to the appropriate order, in terms of the blade geometry.

"Tip effects are considered by extending the theory of Evvard and Krasilshchikova. The present investigation, however, is restricted to those planforms for which no vortex sheet appears off the tip. For points in the tip region, the potential is obtained through the appropriate distribution of 'modified' sources in the upwash region off the tip. By transforming to a curvilinear, non-orthogonal coordinate system coincident with the modified Mach lines described by the infinities of the potential, an integral equation for the required source distribution in the upwash region is derived. Without having to solve this equation, it is shown that the potential for a point in the tip region can be obtained in terms of an integration of known source distributions over the blade surface only.

"The case of a twisted flat plate of particular planform is treated, and a sample calculation is made of the pressure distribution at selected radial positions within the non-communicating portion of the blade, as well as over the entire tip region.

"Though this analysis is carried out explicitly for the supersonic propeller, it could also be extended to calculate various rotary derivatives for highspeed flight vehicles."

3293:

Couchet, G. Efforts d'un fluide incompressible sur un profil oscillant autour d'une incidence non nulle. O. N. E. R. A. Publ. no. 96 (1959), 19 pp.

L'auteur détermine les efforts globaux agissant sur un profil, type Joukowski, lorsque celui-ci, supposé rigide, exécute de petites oscillations, autour de son mouvement moyen qui est une translation de vitesse uniforme sous l'angle d'attaque α. La présente publication se distingue des travaux déjà anciens et classiques [voir par exemple: Donovan and Lawrence (Editors), *Aerodynamic components of aircraft at high speeds*, Princeton Univ. Press, Princeton, N.J., 1957; MR **18**, 844; Sections F 5–8] en ce que des termes non linéaires sont pris en compte; ceux-ci sont relatifs à l'épaisseur p du profil et à l'angle d'attaque α, mais le mouvement oscillatoire est linéarisé. Les résultats s'expriment par deux formules donnant la force complexe et le moment complexe, ce dernier évalué par rapport au centre du profil. A partir des formules données, l'on pourra calculer les dérivées usuelles de flottement. Chacune des dérivées comprend quatre termes:

$$D = D^{(0)}+pD^{(1)}+\alpha D^{(2)}+\alpha^2 D^{(3)}.$$

Il n'y a pas de terme en α dans les moments non stationnaires. Les $D^{(0)}$ sont identiques à ceux que donne la théorie classique du profil mince; leurs expressions en fonction de la fréquence réduite font intervenir les fonctions de Hankel de seconde espèce, qui se retrouvent également dans les $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, sous forme de combinaisons rationnelles ou de quadratures simples ou doubles; c'est à dire que les résultats ne sont pas directement utilisables sans une tabulation préalable. Le rapporteur attire l'attention sur le fait que les termes d'épaisseur en p sont relatifs à une forme très particulière du profil. La méthode de dérivation est directe: représentation conforme sur le cercle; sillage tourbillonnaire le long de la ligne de courant stationnaire s'échappant du bord de fuite, d'intensité régie par la condition de Joukowski; linéarisation effectuée sur les formules exactes.

J. P. Guiraud (Meudon)

3294:

Low, A. H.; Woods, L. C. Unsteady flow through a cascade of aerofoils. J. Austral. Math. Soc. **1** (1959/61), 220–232.

The methods of an earlier paper [Woods, Proc. Roy. Soc. London Ser. A **228** (1955), 50–65; MR **16**, 763] are extended to the case wherein the stream velocity is fixed in direction but its magnitude is variable with time. The methods used are much the same. By means of Laplace-transform methods an indicial-lift function is found which is the generalization of the classical Wagner function of single-airfoil theory; it is calculated here only to first order in the chord/gap ratio of the cascade. As examples, the lift and moment on a typical member of the cascade are calculated for fixed incidence and increasing stream speed. Classical results are retrieved as the chord/gap ratio goes to zero.

W. R. Sears (Ithaca, N.Y.)

3295:

Polyahov, N. N.; Pastuhov, A. I. Theory of the lifting surface of the rectangular form. Vestnik Leningrad. Univ. **14** (1959), no. 13, 93–110. (Russian. English summary)

From the authors' summary: "The present paper contains a generalization of the 'vortex' method (Birnbaum, Glauert), which was developed for the case of the wing of the infinite span or the case of a finite span."

3296:

Peters, A. S.; Stoker, J. J. Solitary waves in liquids having non-constant density. Comm. Pure Appl. Math. **13** (1960), 115–164.

Part I treats two superposed layers of different constant density, with a free surface on top. According to the linear shallow-water theory there are then two critical speeds (i.e., limiting speeds of waves of very long wavelength). It appears that corresponding to these there are two types of solitary wave. One type has characteristics similar to the ordinary solitary wave with a maximum amplitude of motion at the free surface, the other type is an internal wave which at the interface may be a wave either of elevation or of depression depending on the ratios of depth and of density in the two layers.

Part II treats a fluid with a vertical density distribution which decreases exponentially going upwards; the linear theory then furnishes an infinite discrete spectrum of critical speeds, with 0 as the only limit point. With each speed there is associated a solitary wave; the extrema of these are associated with different levels in the fluid which increase in number and begin nearer the bottom as the corresponding critical speed is taken smaller. Their properties are discussed in considerable detail.

In all this work only the leading terms are found, corresponding to the expressions for the solitary wave found by Rayleigh and Boussinesq.

F. Ursell (Cambridge, England)

3297:

Garabedian, Paul R. Numerical estimates of contraction and drag coefficients. Boundary problems in differential equations, pp. 11–18. Univ. of Wisconsin Press, Madison, 1960.

The author seeks the contraction coefficient C_λ for a radially symmetric jet, entering $x>0$ through an orifice in $x=0$, in a space of $\lambda+2$ dimensions, where the stream function satisfies $\psi_{xx}+\psi_{yy}=\lambda y^{-1}\psi_y$, $\mathrm{Re}\,\lambda>-1$. The classical result is $C_0=0.61$, and Trefftz has conjectured that this result should apply to an orifice of any shape. The author disproves this conjecture by deducing the limit $C_\lambda\to 1$, $\lambda\to -1$. He also solves the problem for $\lambda=\infty$, interpolates the results for $\lambda=-1, 0, \infty$ through a power series in $\delta=\lambda/(\lambda+2)$, and calculates the linear term in δ exactly. He concludes that $C_1=0.58$ for an axially symmetric jet and cites experimental support for this result.

J. W. Miles (Los Angeles, Calif.)

3298:

Kumai, Toyoji. On the virtual inertia coefficients for the vertical vibration of ships. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **7** (1959), 245–258.

Author's summary: "An investigation is made of the virtual inertia coefficients for the vertical vibration of ships by means of the strip method, taking the effects of the virtual rotatory inertia of the water and the three-dimensional motion into account. For estimating the three-dimensional correction factors of the virtual inertia coefficients for the vertical vibration of ships, experimental studies on the vibration of cylindrical solid models with several kinds of sectional forms are carried out and their results are taken into account. The analyses of the three-dimensional motion of the circular and elliptic cylinders on the water, and the experimental studies of the measurements of the virtual inertia coefficients for the vibrations of the solid beams and the tanker model are presented in Appendices I and II respectively in the present paper."

3299:

Kumai, Toyoji. Added mass moment of inertia induced by torsional vibration of ships. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **7** (1959), 233–243.

Author's summary: "An investigation is made of the added mass moment of inertia induced by the rotational motion of prisms having sections similar to the hull sections on water to obtain the information to estimate the natural frequency of the torsional vibration of ships. Special consideration is paid to the effect of the draught upon the added mass moment of inertia in the present study. Experimental studies using prismatic models with the sectional forms of the main stations of the hull of a tanker are also carried out confirming the theoretical calculation. As a result of the present investigation, a convenient formula for estimation of the added mass moment of inertia induced by the torsional vibration of a tanker, for an example, is deduced for practical use."

3300:

Gaillard, P. Des oscillations non linéaires des eaux portuaires. Houille Blanche **15** (1960), 164–172. (English summary)

A wave maker is placed at one end of a rectangular channel, while the other end opens onto a rectangular basin (the "harbour"). The problem is to determine the oscillations of the water in the basin caused by swells generated by the wave maker. A formal method of solution is given for both the first and second order approximations. Even for the first order problem, however, the method involves the solution of an infinite system of linear equations, which appears very formidable to handle, either theoretically or numerically. A simpler but "less precise" method due to Takano is also described, and some results of numerical calculations of Takano are presented. On the basis of his second order theory the author indicates the following non-linear effect. Two or more distinct forcing frequencies in the spectrum of the incident swell may produce a seiche of much smaller frequency in the basin. This effect has been observed in harbours, and may account for the creation of seiches which endanger the moorings of ships.

D. H. Hyers (Los Angeles, Calif.)

3301:

van Dantzig, D.; Lauwerier, H. A. The North Sea problem. I. General considerations concerning the hydrodynamical problem of the motion of the North Sea. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 170–180.

The equations of motion are derived in the form

$$\left(\frac{\partial}{\partial t}+\lambda\right)u-\Omega v+gh\frac{\partial\zeta}{\partial x} = U,$$

$$\left(\frac{\partial}{\partial t}+\lambda\right)v+\Omega u+gh\frac{\partial\zeta}{\partial y} = V,$$

where ζ is the elevation, (u, v) is the velocity, and λ is constant. The terms λu, λv on the left express the effect of bottom friction, the terms on the right the effect of surface stresses. Two different boundary conditions are applied along different parts of the boundary. Along the coastal part the normal velocity vanishes, along the oceanic part the elevation vanishes. The velocity components are now eliminated from the equations of motion and of continuity, and the resulting equations for ζ are subjected to a Laplace transformation with respect to the time. The transformed equations can be solved if a Green's function can be found. The Green's function is found which satisfies

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2\right)G(x, y; x_0, y_0) = 0$$

in the half-plane $y>0$; which satisfies

$$\left(\cos\gamma\frac{\partial}{\partial y}-\sin\gamma\frac{\partial}{\partial x}\right)G = 0$$

on $y=0$; which vanishes at infinity; and which has a logarithmic singularity at (x_0, y_0).

F. Ursell (Cambridge, England)

3302:

Debye, Peter; Daen, Jerome. Stability considerations on nonviscous jets exhibiting surface or body tension. Phys. Fluids **2** (1959), 416–421.

Authors' summary: "The question of the range of ejected liquid streams is connected with the relation of the stability of the jet motion to an initial infinitesimal disturbance. This is investigated as a theoretical problem for the planar and cylindrical cases neglecting any fluid viscosity. In addition, the stabilizing influence of either a surface or 'body tension' is calculated. Comparison of the theory with some published data is found to yield qualitative and semiquantitative agreement. The possibility of applying this theory to the measurement of dynamic surface tensions is suggested."

3303:

Taylor, Geoffrey; Saffman, P. G. A note on the motion of bubbles in a Hele-Shaw cell and porous medium. Quart. J. Mech. Appl. Math. **12** (1959), 265–279.

The motion of a bubble in a stream of viscous liquid between two parallel plane plates close together is considered; the viscosity of the liquid in the bubble is different from that of the liquid in the main stream and is ignored in part of the work. Hele-Shaw equations in terms of the velocity averaged through the distance between the plates are used; these are in the same form as those for two-dimensional flow in a porous medium. For a bubble with area of section S (and thickness equal to the distance between the plates) a single relation is formed between the velocity of the bubble, its maximum breadth and S. For given S and main-stream velocity there is accordingly an infinity of solutions; the physical significance of this is considered. Conditions for the motion of large (semi-infinite) bubbles, not symmetrical about the line of main flow are given; this continues previous work by the authors [Saffman and Taylor, Proc. Roy. Soc. London. Ser. A **245** (1958), 312–329; MR **20** #3697] on large symmetrical bubbles.

W. R. Dean (London)

3304:

Džorbenadze, N. P. On non-stationary flow of a viscous fluid between parallel porous walls. Soobšč. Akad. Nauk Gruzin. SSR **22** (1959), 657–663. (Russian)

3305:

Brenner, Howard. Dissipation of energy due to solid particles suspended in a viscous liquid. Phys. Fluids **1** (1958), 338–346.

The additional mechanical energy dissipation due to solid particles suspended in a viscous incompressible fluid in laminar motion is computed. It is assumed that whether or not the suspended particles are present the motion is governed by the Stokes approximations. Then by use of a reciprocal theorem connecting Stokes flows, the additional dissipation is expressed in terms of surface integrals over the particle surfaces alone. These integrals are evaluated for the case of spherical particles.

The theory is applied to deduce the pressure drop due to the passage of a liquid through a bed of fluidized solids. A suitable general definition of the apparent viscosity of a suspension leads to Einstein's result for the viscosity of a dilute random suspension of spherical particles, and an inequality is deduced to show that for a dilute random suspension of particles of any shape the viscosity is always increased.

H. C. Levey (Perth)

3306:

Morton, B. R. Laminar convection in uniformly heated horizontal pipes at low Rayleigh numbers. Quart. J. Mech. Appl. Math. **12** (1959), 410–420.

In liquid forced under a pressure gradient γ through a horizontal pipe with walls maintained at a temperature gradient τ the buoyancy forces superpose a secondary flow on the main Poiseuille flow. A series solution is found for a pipe with a circular cross-section in terms of a non-dimensional constant AB which contains the factor $g\beta\gamma\tau$, β being the coefficient of expansion of the liquid. By successive approximation terms up to the order $(AB)^2$ are found. If $AB=3000$, it is shown that an error of order 10% is made by neglecting the buoyancy forces.

W. R. Dean (London)

3307:

Hocking, L. M. The Oseen flow past a circular disk. Quart. J. Mech. Appl. Math. **12** (1959), 464–475.

Using Oseen's equations the steady flow of viscous liquid past a circular disc, with its plane normal to the stream, is found by taking distributions of doublets and of singularities of the next higher order over the area of the disc. Solutions are found for low and for high Reynolds numbers.

W. R. Dean (London)

3308:

Pearson, J. R. A. The instability of uniform viscous flow under rollers and spreaders. J. Fluid. Mech. **7** (1960), 481–500. (1 plate)

When a thin film of viscous fluid is produced by passing it through a small gap between a roller or spreader and a flat plate, it often presents a waved, or ribbed, surface. The author investigates this effect by the lubrication theory including the surface tension. Some approximate boundary conditions are physically argued and proposed. Then a case of a spreader in the form of a wide-angled wedge is solved. It is found that the most unstable values of the disturbance wave numbers are related to the dimensionless parameter $T/\mu U$ or surface tension/viscosity $\times$ velocity. The result is compared to experimental observations. It shows a general agreement except at small values of $T/\mu U$. *L. N. Tao* (Chicago, Ill.)

3309:

Carrier, G. F.; Miles, J. W. On the annular damper for a freely precessing gyroscope. J. Appl. Mech. **27** (1960), 237–240.

An interesting technical application of viscous-fluid theory: a tube running round the rim of a gyroscope is partly filled by liquid, and an estimate is made of the rate at which viscous action damps a wobbling motion of the gyroscope. *T. M. Cherry* (Melbourne)

3310:

Ting, Lu. Boundary layer over a flat plate in presence of shear flow. Phys. Fluids **3** (1960), 78–81.

The velocity in the shear flow is given by $u=U_0+\omega y$. The author considers flows with large vorticity numbers $\Omega=\omega\delta/U_0$, where δ is the "length scale" in y direction inside the boundary layer compared to L, the length scale for the flow outside the layer. The conditions for transition of the boundary layer flow into the exterior flow are carefully discussed. For large Ω there exists a self-similar solution whose streamfunction can be expressed as $\psi=(x/L)^{1/3}f(\eta)$ with $\eta=y(x/L)^{-1/3}$. $f''(\eta)$ determines the shear stress due to viscosity; it has the value 0.7866 at the wall. In the last part of the paper approximate solutions for moderate vorticity numbers are given. *I. Flügge-Lotz* (Stanford, Calif.)

3311:

Yang, Kwang-Tzu. Possible similarity solutions for laminar free convection on vertical plates and cylinders. J. Appl. Mech. **27** (1960), 230–236.

Conditions are derived under which similarity solutions of the unsteady boundary-layer equations for this problem are possible in the case of constant fluid properties. In addition to various steady cases, which are essentially covered by the available similarity solutions in the literature, there are several new cases involving unsteady conditions. *D. W. Dunn* (Ottawa, Ont.)

3312:

Poots, G. A solution of the compressible laminar boundary layer equations with heat transfer and adverse pressure gradient. Quart. J. Mech. Appl. Math. **13** (1960), 57–84.

This paper presents an important addition to the library of exact solutions of the boundary layer equations. The author assumes that in the gas the viscosity is proportional to the temperature and the Prandtl number is unity. Further, the main-stream is such that in the equivalent incompressible plane the main-stream is proportional to $1-x/8$ where x measures distance along the boundary. Finally, the wall is heated so that heat is transferred from it to the fluid. Thus the situation envisaged is complementary to that studied at the National Physical Laboratory and reported by Curle [Proc. Roy. Soc. London. Ser. A **249** (1959), 206–224; MR **20** #6887] in which the wall was cooled.

The solution was found by numerical integration using series expansions in the range $0<x<0.2$ and a step-by-step integration in the range $x>0.2$. Separation was found to occur very close to $x=0.60$ showing that the effect of heating the wall is to encourage separation. {Without heat transfer separation would occur at $x=0.96$.} The author does not commit himself on the question of singularities at separation but his solution does not appear to imply that they occur.

The results are compared with Tani's [J. Aeronaut. Sci. **21** (1954), 487–495; 504; MR **15**, 999] approximate method using the momentum, energy and thermal integrals of the equations. The agreement is excellent, the discrepancies in the skin friction and heat transfer being at worst only about 4%. *K. Stewartson* (Durham)

3313:

Yuan, S. W. ★Cooling by protective fluid films. Turbulent flows and heat transfer (edited by C. C. Lin), pp. 428–488. High Speed Aerodynamics and Jet Propulsion, Vol. V. Princeton University Press, Princeton, N.J., 1959. xv + 549 pp. (2 plates) $15.00.

The author presents a review of the fundamental aspects of the cooling of surfaces exposed to high-temperature gas flow by injection of a cooling fluid. Transpiration cooling, in which the coolant passes through a porous wall into the external hot flow, is emphasized, and the injected fluid is assumed to be the same as the external fluid. The theoretical results are developed for laminar and turbulent boundary layers and pipe flow. A brief discussion is also given of film cooling, in which the cooling fluid is discharged through slots or orifices into the flowing hot gas. References to papers published as late as 1958 are included, and the article appears to be a comprehensive and useful review of research in the subject up to that date. *D. W. Dunn* (Ottawa, Ont.)

3314:

Kulonen, G. A. The method of successive approximations of M. E. Švetc applied to the calculation of boundary layer in compressible gas. Vestnik Leningrad. Univ. **15** (1960), no. 1, 123–131. (Russian. English summary)

Author's summary: "The method of successive approximations is used for solving the laminar boundary layer problem in a compressible gas on the porous flat plate under the arbitrary change of the vertical velocity component on the wall. The method is illustrated by examples. When $v_w\sim 1/\sqrt{x}$ a comparison with the exact solution is given."

3315:

Gribben, R. J. Laminar boundary layer with suction and injection. Phys. Fluids **2** (1959), 305–318.

The transformation rules connecting compressible and incompressible, axi-symmetric and plane, boundary layers are extended to include the effects of suction and injection. The results are applied to the supersonic flows over a cone and near the tip of a general slender body of revolution.

{A serious omission from the discussion is whether the flows considered are realistic. For example in the study of the flow over a cone it is assumed that the inviscid flow is tangential to the surface of the cone. However, if the cone has a porous surface to permit injection, there will be a tendency for the oncoming fluid to penetrate into the cone. If this occurs the ratio of the actual normal velocity at the surface of the cone to the main-stream velocity is formally of order one whereas for a boundary layer to be relevant this ratio should be of the order of the inverse of the square root of the Reynolds number. For this reason it is difficult, if not impossible, to envisage an experiment which meets the requirements of the theory.}

K. Stewartson (Durham)

3316:

Merkulov, V. I. Heat exchange in plane laminar flow of a viscous fluid. Prikl. Mat. Meh. **23** (1959), 581–582; erratum, **24** (1960), no. 2, inside back cover (Russian); translated as J. Appl. Math. Mech. **23**, 819–822; erratum, **24**, 573.

The author considers the problem of heat transfer between a solid body with specified surface temperature and a stream of incompressible fluid with constant heat conductivity and constant temperature at infinity, the velocity field being assumed to be known. After a transformation of the independent variables from the space variables to the velocity potential and stream function of the ideal inviscid flow about the same body, the problem is reduced to an integral equation in which the non-homogeneous term gives the solution for the temperature field in an inviscid fluid. A proof of the existence of a unique solution in a viscous fluid is briefly outlined.

D. W. Dunn (Ottawa, Ont.)

3317:

Bourne, D. E. A note on the approximate calculation of the temperature distribution in an incompressible laminar boundary layer over a heated plane surface. Quart. J. Mech. Appl. Math. **12** (1959), 337–339.

In this paper it is shown that the temperature distribution T can be expressed as

$$T_0+\sum_{n=0}^{\infty} a_n x^n X_n(\xi),$$

where a_n are constants, x is the distance from the leading edge of the surface and $X_n(\xi)$ is a Whittaker function of the type $W_{k,m}(\xi)$. *L. J. Slater* (Cambridge, England)

3318:

Chandrasekhar, S. The hydrodynamic stability of inviscid flow between coaxial cylinders. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 137–141.

This paper presents a generalization, to include the effects of an axial flow, of Rayleigh's criterion on the instability of a fluid in differential rotation. It is first shown that in the case of a purely axial flow the condition for instability is that the quantity $rd(r^{-1}dW/dr)/dr$ change sign—this result being a generalization of Rayleigh's inflection-point theorem for plane flows. In the general case, however, there is a coupling between the radial and transverse perturbations of the velocity and the criterion for stability is that $\Omega r^{-1}d(r^2\Omega)/dr$ be positive. Thus, in the presence of rotation, Rayleigh's criterion remains valid irrespective of W. *W. H. Reid* (Providence, R.I.)

3319:

Chandrasekhar, S. The hydrodynamic stability of viscid flow between coaxial cylinders. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 141–143.

In this paper the author considers the effect of an axial flow on the stability of viscous flow between rotating cylinders. In the "small-gap" approximation, the mean angular velocity has a linear variation while the axial velocity has a parabolic form. A very considerable simplification of the disturbance equations is achieved by assuming that both of these mean velocities can be replaced by their averages. Averaging of the angular velocity is known to be a good approximation if the cylinders rotate in the same direction but averaging of the axial velocity can only be expected to be a good approximation if the axial Reynolds number R is small. Numerical results are given for values of R up to 100 for which the Taylor number is an increasing function of R. For sufficiently large values of R, the axial flow must become unstable through the Tollmien-Schlichting mechanism; this would explain an apparent anomaly in the inviscid analysis [see preceding review] since the axial flow considered here is stable on inviscid theory.

W. H. Reid (Providence, R.I.)

3320:

Sato, Hiroshi. The stability and transition of a two-dimensional jet. J. Fluid Mech. **7** (1960), 53–80. (4 plates)

The paper begins with a presentation of the results of an experimental investigation, and concludes with a theoretical study of the stability of the two-dimensional laminar jet based on the Orr-Sommerfeld equation in the inviscid limit. The velocity profile is represented by a two-parameter interpolation formula, which includes the theoretical profile for a fully-developed jet and is also capable of representing closely the initial profiles observed experimentally in the developing flow. With a proper choice of the velocity profile, numerical calculations of the eigenvalues and eigenfunctions are in good agreement with the experimental results. *D. W. Dunn* (Ottawa, Ont.)

3321:

Chang, C. T. Dynamic instability of accelerated fluids. Phys. Fluids **2** (1959), 656–663.

An inviscid liquid fills the half-space $y<0$ in a reference frame subjected to a constant acceleration $-g^*$ in the negative-y direction, and the free surface is given the periodic displacement $a\cos(kx)$ at $t=0$. According to linearized theory, the surface motion will grow exponentially for $k<k_c$ and oscillate harmonically for $k>k_c$, where $k_c=(-g^*/T_1)^{1/2}$, $T_1=$specific surface tension. The author

formulates the non-linear, initial-value problem through an expansion in powers of the amplitude a and gives results based on the retention of terms through a^3. His results for $k<k_c$ agree with the observed growth pattern of sharpened crests and flattened troughs. He also finds an oscillatory growth of $O(a^3t)$ for $k>k_c$ and cites experimental evidence in qualitative support. As he points out in a footnote, however, his theory cannot distinguish such a growth from an oscillatory solution of the form $a\cos\{(\omega^2-\omega_c^2)^{1/2}t[1+O(a^2)]\}+O(a^5)$, while the experimental results may be associated with non-sinusoidal initial disturbances. This reviewer, bearing in mind the well-known secular instability of perturbation theory [Rayleigh, *The theory of sound*, Dover, New York, 1945; MR 7, 500; § 67] and the oft-cited ability of a fine-mesh cloth to hold water, believes that the author's results for $k>k_c$ require further investigation before they can be accepted unequivocally. *J. W. Miles* (Los Angeles, Calif.)

3322:

Tao, L. N. A theory of lubrication with turbulent flow and its application to slider bearings. J. Appl. Mech. **27** (1960), 1–4.

Author's summary: "The governing equation of turbulent lubrication in three dimensions, equivalent to the Reynolds equation of laminar lubrication, is derived. The problem of a slider bearing with no side leakage is then analyzed. An exact solution is found in closed form. Bearing characteristics are also established. It is found that the Reynolds number is an important parameter in the problem of turbulent lubrication. Furthermore, it is shown that the laminar lubrication may be considered as the special case of the present study. A numerical example is also included."

3323:

Davydov, B. I. On the statistical dynamics of an incompressible turbulent fluid. Dokl. Akad. Nauk SSSR **127** (1959), 768–771 (Russian); translated as Soviet Physics. Dokl. **4** (1960), 769–772.

Turbulent shear flow is discussed by writing down equations for the mean velocity, U_i, for the Reynolds stress tensor, R_{ij}, and for the transport tensor, S_{ijk}, all velocities measured at the same point in space. Observing that the pressure-velocity terms in the equation for R_{ij} are analogous with the collision terms in the Boltzmann equation, it is assumed that

$$(1)\quad \overline{q\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)} = -\frac{\beta Q}{R}\,(R_{ij}-\tfrac{1}{3}\delta_{ij}R)-B_{ij}+\tfrac{1}{3}\delta_{ij}B_{kk},$$

where Q is the energy dissipation and B_{ij} is a tensor introduced here to allow anisotropy due to nearness of a solid wall. In the equation for S_{ijk}, the distributions of velocity components are assumed to be quasi-normal and, extending the analogy of (1),

$$(2)\quad \overline{\frac{\partial q}{\partial x_i}v_jv_k}+\overline{\frac{\partial q}{\partial x_j}v_iv_k}+\overline{\frac{\partial q}{\partial x_k}v_iv_j} = \frac{\beta_1 Q}{R}S_{ijk}.$$

These assumptions are not derivable from the equations of motion and introduce an element of irreversibility not present in the original equations.

A. A. Townsend (Cambridge, England)

3324:

Davydov, B. I. On the statistical theory of turbulence. Dokl. Akad. Nauk SSSR **127** (1959), 980–982 (Russian); translated as Soviet Physics. Dokl. **4** (1960), 779–781.

The equations for the quantities, U_i, R_{ij}, S_{ijk}, from the previous paper [see preceding review] are used to consider flow in a turbulent boundary layer. Assuming that net energy transport is negligible, $Q=-R_{xy}U_x'$, and consistency of the equations is secured if the only non-zero component of the introduced tensor is

$$(1)\quad B_{yy}=\tfrac{1}{2}R_{xy}U_x'(2-\beta+3\beta^2R_{xy}^2/R^2).$$

A knowledge of the energy dissipation is necessary for the determination of the form of the mean velocity distribution, and an equation for Q is obtained by postulating an analogy between the behaviour of Q and of the energy density $\frac{1}{2}R$ and comparing this equation with observations of the decay of homogeneous turbulence. Then,

$$(2)\quad (QS_y)'+(2-\gamma)QR_{xy}U_x'+2Q^2=0$$

and, in the constant-stress layer where S_y and R_{xy} are nearly constant, this leads to the logarithmic profile. A comparison with the measurements of Laufer gives satisfactory agreement for the distributions of U_x and R with a suitable choice of the arbitrary constants, γ and β.

A. A. Townsend (Cambridge, England)

3325:

Reid, W. H. One-dimensional equilibrium spectra in isotropic turbulence. Phys. Fluids **3** (1960), 72–77.

La théorie du transfert d'énergie turbulente d'un nombre d'onde à un autre a fait l'objet d'hypothèses de Heisenberg [Z. Physik **124** (1948), 628–657; MR **11**, 63], Kovasznay [J. Aeronaut. Sci. **15** (1948), 745–753; MR **10**, 412], Obukhoff [C.R. (Doklady) Acad. Sci. URSS (N.S.) **32** (1941), 19–21; MR **3**, 221]. Seule la théorie de Heisenberg a donné lieu à des vérifications expérimentales, satisfaisantes, mais peu précises. L'auteur se propose d'étudier les trois théories dans des conditions où la précision des mesures soit meilleure et la comparaison plus sensible. Il ramène les fonctions spectrales à des formes réduites sans dimensions, dépendant d'un paramètre que l'éxperience seule permet de choisir. L'emploi du facteur de distorsion (skewness) donne des résultats qui dépendent beaucoup du nombre de Reynolds. Au contraire, pour les grands nombres d'onde, le spectre de $\partial^3u_1/\partial x^3$ en dépend peu et se prête à la comparaison. Les théories de Heisenberg et Kovasznay permettent un bon ajustement des résultats expérimentaux dans l'intervalle $k\eta<0{,}4$ ($\eta=(\nu^3/\varepsilon)^{1/4}$). La théorie d'Obukhoff est en mauvais accord avec l'expérience. *J. Bass* (Paris)

3326:

Lemehov, E. E. On the length of the mixing path in turbulent motion of incompressible fluid in pipes. Vestnik Leningrad. Univ. **14** (1959), no. 19, 130–134. (Russian. English summary)

Author's summary: "The relations generally used for the length of the mixing path at the turbulent motion of a fluid in pipes cannot be used in studying the turbulent motion of a fluid in flat pipes with asymmetrical distribution of temperature over the section of the pipe. Proceeding from some conjectures a new kind of relation

for the length of the mixing path, that may be applied to round as well as to flat pipes, has been established. The solution is shown to be in good accordance with the experiment in the case of round pipes."

3327:

Fedyaevskiĭ, K. K.; Ginevskiĭ, A. S. Nonstationary turbulent boundary layer on a wing profile and on a figure of rotation. Ž. Tehn. Fiz. **29** (1959), 916–923 (Russian); translated as Soviet Physics. Tech. Phys. **4** (1960), 829–836.

The effect of flow acceleration on the development of a two-dimensional turbulent boundary layer is discussed. As in previous papers by the first author, the square root of the shear stress distribution is expressed as a Pohlhausen-type polynomial in y/δ (δ = boundary layer thickness, y = distance from the wall), and the velocity profile is then found by equating the resulting shear stress to a mixing length expression.

This procedure must always be suspect when applied to a flow such as the present one in which there are no direct experimental results to back it up. Qualitative predictions of an increase or decrease in the overall drag of a circular cylinder in retarded or accelerated motion due to movement forward or back of the point of zero skin friction (which the author identifies with separation, even in a non-stationary flow) do, however, agree with what is found experimentally. *D. A. Spence* (Pasadena, Calif.)

3328:

Uram, Earl M. A method of calculating velocity distribution for turbulent boundary layers in adverse pressure distributions. J. Aerospace Sci. **27** (1960), 659–666, 674.

An empirical defect law of the form $(U_1-U)/U^* = \phi(1)-\phi(\eta)-5.6\ln\eta$ $(\eta>\eta_0)$, where $\phi(\eta)=5.6+a(\eta-\eta_0)+b\exp 20(1-\eta)^2$ $(\eta=U^*y/\nu$, U^* = friction velocity) is shown to provide a working approximation to the velocity distribution in the outer parts of turbulent boundary layers in two-dimensional and conical flows. η_0 is empirically related to the momentum thickness. As is well known $\phi(1)$ depends uniquely on $C_f=2(U^*/U_1)^2$, and it is shown that a and therefore b can also be found from this quantity —although the experimental curves relating a to C_f are different in the two types of flow. The numerical procedure is therefore to compute momentum thickness by one of the standard quadratures, thence to find the skin friction distribution and from this, the velocity profile. Because he has been able to correlate velocity profiles purely in terms of the local skin friction coefficient, the author doubts the significance of Coles's "law of the wake" [J. Fluid Mech. **1** (1956), 191–226; MR **18**, 355], but as figure 5 shows, his approximation is poorest in the region $0.6\delta<y<\delta$, and in the opinion of this reviewer, the physical conclusions drawn from Coles's much more accurate examination of velocity profiles retain their force.

D. A. Spence (Pasadena, Calif.)

3329:

Walker, George K. A particular solution to the turbulent-boundary-layer equations. J. Aerospace Sci. **27** (1960), 715–716.

3330:

Albring, W. Transformation der Impulsgleichung für turbulente Grenzschicht von Rotationskörpern auf ebene Körper. Z. Angew. Math. Mech. **40** (1960), 38–40.

3331:

Chang, Paul K. Differential equation of compressible turbulent skin-friction coefficient around an arbitrary body of revolution at high speed. J. Aero/Space Sci. **27** (1960), 466–467.

3332:

Kosterin, S. I.; Košmarov, Yu. A. The turbulent boundary layer on a flat plate in a uniform stream of a compressible fluid. Ž. Tehn. Fiz. **29** (1959), 906–915 (Russian); translated as Soviet Physics. Tech. Phys. **4** (1960), 819–828.

The author uses Prandtl's mixing length relation $l=ky$ to calculate velocity profiles in the compressible turbulent boundary layer on a flat plate, both laminar and turbulent Prandtl numbers being taken as unity. This approach automatically gives the well-known logarithmic law of the wall, but its validity in the outer part of the boundary layer is open to serious question, and even close to the wall there is some evidence that the density variations should be represented in the mixing length, for example by use of a Dorodnitzyn variable instead of the physical y. A matching analysis of von Karman's type at the edge of the laminar sub-layer is used to give the skin friction coefficient, which is said to fall off with increasing Mach number in good agreement with experimental results. The analysis is very similar throughout to that of Van Driest [J. Aeronaut. Sci. **18** (1951), 145–160, 216; MR **13**, 883].

D. A. Spence (Pasadena, Calif.)

3333:

Powell, Alan. Propagation of a pressure pulse in a compressible flow—coda. J. Acoust. Soc. Amer. **32** (1960), 1116.

Author's summary: "As explained previously [same J. **31** (1959), 1527–1535; MR **22** #2252] the disturbances due to an initially step-like pressure wave progressing down a channel carrying a compressible flow can be analyzed by a multiple reflection method. For the final 'transmitted' and 'reflected' pressures, this yields a power series expansion, successive terms representing the effect of higher order reflections. It is now shown that the coefficients of these series are connected to Euler's and Bernoulli's numbers, respectively, and convenient expressions for the coefficients are given. When n is not small, the coefficient $c^n \simeq 2(2/\pi)^{n+1}$. The infinite power series are shown to have sums simply like sech x and tanh x, respectively. This provides an easy means of numerical evaluation, and gives simple criteria for the accuracy of the approach using the leading term of each of the series. The results when reflections of all orders are taken into account are shown to be analytically identical to those of the 'before' and 'after' steady flow method."

3334:

Koldobskaya, T. G. Non-stationary motion problem near to automodel. Vestnik Leningrad. Univ. **15** (1960), no. 1, 111–122. (Russian. English summary)

Author's summary: "Plane and axisymmetrical non-stationary gas motions near to automodel are considered. A method of linearization near the known automodel solution is used. The method can be applied to a problem of plane shock wave flow round the body near to a wedge or a cone."

3335:

Miheev, A. S. Equations of gas dynamics for the case of axial symmetry. Vestnik Leningrad. Univ. **14** (1959), no. 13, 142–144. (Russian. English summary)

Author's summary: "Equations of the established, axially symmetrical motion of an ideal compressible fluid have been derived, the dependence between pressure, density and function of the current being arbitrary.

"An example of the possibility of reducing a vortex movement to a potential one has been shown."

3336:

Trošin, V. I. Gas flow through an opening in a channel wall. Prikl. Mat. Meh. **23** (1959), 944–947 (Russian); translated as J. Appl. Math. Mech. **23**, 1340–1345.

The problem is two-dimensional, and the conditions are supposed everywhere subsonic. The solution is set up in the hodograph plane by means of three series for the stream function, valid in different regions, and the coefficients are determined from continuity and boundary conditions. *T. M. Cherry* (Melbourne)

3337:

Sidorov, A. F. Nonstationary potential flow of a polytropic gas with degenerate hodograph. Prikl. Mat. Meh. **23** (1959), 940–943 (Russian); translated as J. Appl. Math. Mech. **23**, 1334–1339.

(1) With the aid of a Legendre potential, the equations of unsteady irrotational flow of a polytropic gas are transformed to hodograph variables; it appears that the treatment implicitly supposes that the speed is independent of t. (2) A similar transformation is made for the case where the space coordinates and u_3 are functions of t and two velocity coordinates u_1, u_2 only. No applications are developed. *T. M. Cherry* (Melbourne)

3338:

Cremer, Hubert; Kolberg, Franz. Windkanalkorrekturen angestellter Körper bei kompressibler Unterschallströmung. Z. Angew. Math. Mech. **40** (1960), 65–74.

The authors find corrections in the velocity components due to the effect of the walls for a subsonic flow about a model placed in a wind tunnel of circular cross section. The calculations are based upon the potential equation of compressible subsonic flow in Prandtl's linearized form, together with boundary conditions for both closed and open test sections. The model is replaced by a source-sink combination whose axis is inclined at an angle α with the axis of the wind tunnel, so that the problem is three-dimensional and not axially symmetric.

The method of solution involves Fourier transforms with respect to the variable along the axis, and the solutions are exhibited as Fourier series in the angular variable, whose coefficients are improper integrals containing certain combinations of Bessel functions. From these series solutions approximate formulas are derived which are valid for small values of the parameter $L \sin \alpha$, where L denotes the ratio of the source-sink distance to the diameter of the wind tunnel. *D. H. Hyers* (Berkeley, Calif.)

3339:

Rabinovič, B. I. On small harmonic oscillations of a cylindrical shell along the axis of which an ideal gas flows with supersonic velocity. Prikl. Mat. Meh. **23** (1959), 879–884 (Russian); translated as J. Appl. Math. Mech. **23**, 1255–1262.

Author's summary: "A circular cylindrical shell is considered to have a flat bottom at one end where a system of uniformly distributed supersonic sources is located. The other end of the shell is open and through it flows a uniform supersonic stream of an ideal gas originating at the bottom. On the assumption that the shell performs small harmonic oscillations in a certain plane, the dynamic interaction between the gas and the shell walls is investigated. The gas compressibility leads to the appearance of non-stationary forces whose role in the general scheme depends upon the Strouhal number; in other words, the principal vector of the gas-dynamic forces manifests itself during the shell oscillation as displacement and rotation relative to the longitudinal axis." *G. N. Lance* (Winfrith, Dorset)

3340:

Miles, John W. On supersonic flutter of long panels. J. Aero/Space Sci. **27** (1960), 476.

3341:

Lick, Wilbert. Inviscid flow of a reacting mixture of gases around a blunt body. J. Fluid Mech. **7** (1960), 128–144.

A blunt body is fixed in an inviscid and chemically inert mixture of gases moving with supersonic speed. It is assumed that conditions across the detached shock wave can be specified by the conventional equations but that thereafter the temperature of the gas is sufficiently high for dissociation and recombination to occur. The governing equations are set out and, assuming that the shock is given, a numerical step-by-step method (in spite of its attendant difficulties) described for computing the flow field behind the shock. Four numerical examples are worked out for a Mach number at infinity of 14 and for conditions obtaining at 100,000 ft. *K. Stewartson* (Durham)

3342:

Wachtell, G. P.; Carfagno, S. P. Supersonic flow in a tube with longitudinal slots. Phys. Fluids **2** (1959), 521–526.

A differential equation for the supersonic flow in a longitudinally slotted tube is derived on the basis of the usual assumptions of one-dimensional homentropic flow with the additional assumption of critical flow through the slots.

The approximate integration of this equation is discussed when the cross-sectional area and total slot width are arbitrary functions of the downstream distance, and

results are given for the case when these parameters are constant. Comparison with experiment shows the need for an additional correction factor which is explained as due to the boundary layer effectively decreasing the slot width and throat area. *H. C. Levey* (Perth)

3343:

Kogan, Abraham. On supersonic flow past thick airfoils. J. Aero/Space Sci. **27** (1960), 504–508, 516.

The pressure distribution along the entire length of a supersonic aerofoil is obtained. The assumption is made that the shock waves are attached to the leading edge of the aerofoil. An iterative method is used to calculate successive terms in series expansions of the co-ordinates and velocity components. The expansion parameter is conveniently chosen to be the Crocco stream function. The method is such that the accuracy, for a given number of iterations, is better for thick than for thin aerofoils. The results obtained are compared with those obtained by the method of characteristics and the shock expansion method. *G. N. Lance* (Winfrith, Dorset)

3344:

Stanišić, M. M. On the unsteady motion of a delta wing in supersonic flight. J. Aero/Space Sci. **27** (1960), 399–400.

The author uses the Volterra-Green method to formulate the acceleration potential for the unsteady motion of a rigid delta wing in supersonic flow.

G. N. Lance (Winfrith, Dorset)

3345:

Landahl, Mårten; Drougge, Georg; Beane, Beverly. Theoretical and experimental investigation of second-order supersonic wing-body interference. J. Aerospace Sci. **27** (1960), 694–702.

Authors' summary: "Approximate second-order solutions for the supersonic flow around wing-body combinations are calculated, using two different theoretical models. In the first, the spanwise curvature of the body field is assumed small and the wing sweep small in comparison with that of the Mach cone. In the second, two perpendicular intersecting two-dimensional fields are considered. The analysis is restricted to such high Mach numbers that $M^{-2} \ll 1$, and an approximate formula common to the two models is then found for the second-order interference term. This formula can also be used to correct experimental pressure distributions for the effect of nonuniformities in the wind-tunnel flow."

3346:

Herring, T. K. The boundary layer near the stagnation point in hypersonic flow past a sphere. J. Fluid Mech. **7** (1960), 257–272.

Cet article représente une contribution au problème du transfert de chaleur en atmosphère modérément raréfiée. Seul le voisinage immédiat du point d'arrêt est étudié par une méthode qui prolonge celle de Lighthill [same J. **2** (1957), 1–32; MR **19**, 352], mais en remplaçant les équations des fluides parfaits par les équations de la couche limite. Les conditions aux limites retenues sont la condition de non glissement et de paroi froide sur l'obstacle, et les conditions de Rankine-Hugoniot sur l'onde de choc. Probstein et Kemp [J. Aero/Space Sci. **27** (1960), 174–192; MR **22** #1267] ont fait la même chose, mais en retenant les équations des fluides visqueux (viscous layer régime). Les résultats, détaillés, sont présentés, soit sous forme de tableaux, soit sous forme de courbes, pour diverses valeurs de γ (l'auteur travaille en gaz parfait), du nombre de Mach, et du nombre de Reynolds (500 à 40.000, basé sur les conditions amont et le rayon de la sphère). La distance de détachement, assez curieusement, croit avec le nombre de Reynolds. La variation du coefficient de frottement est indiquée, mais rien n'est dit sur le transfert de chaleur, bien que les informations nécessaires paraissent figurer dans les tables.

J. P. Guiraud (Meudon)

3347:

Makofski, Robert A. On the use of the boundary-layer equations in the hypersonic viscous-layer regime. J. Aero/Space Sci. **27** (1960), 468–469.

3348:

Strand, T. Minimum wing wave drag with volume constraint. J. Aerospace Sci. **27** (1960), 615–619.

Author's summary: "A numerical method is developed for calculating the minimum thickness drag for a given wing planform and volume using linearized supersonic flow theory. The corresponding optimum volume distribution is also determined. The results show that considerable drag reduction is possible by improved volume distribution."

3349:

Morkovin, M. V. Note on the assessment of flow disturbances at a blunt body traveling at supersonic speeds owing to flow disturbances in free stream. J. Appl. Mech. **27** (1960), 223–229.

A linearised analysis is presented to obtain the amplification factor for disturbances of a field of flow of a perfect gas between a plane shock and a plane boundary. Disturbances arising at the body and in the main stream ahead of the shock are considered. It is shown that no resonant growth of disturbances can occur and plausible arguments are given to suggest that this result holds for supersonic flow of a real gas past a bluff body.

J. J. Mahony (Sydney)

3350:

Riley, N. Interaction of a shock wave with a mixing region. J. Fluid Mech. **7** (1960), 321–339. (1 plate)

The problem investigated is that of calculating the steady flow when a weak plane discontinuity (either shock or expansion wave) is reflected from a mixing region which forms a plane boundary between a uniform supersonic stream and fluid at rest. The mixing region is represented as a shear flow, with arbitrary profile, parallel to the main flow, and linearized equations are obtained for the disturbances from a parallel flow on the assumption that all other mixing effects can be ignored. The investigation of the asymptotic properties of the Fourier transform of the solution provides a qualitative description of the flow pattern. For complete details of the solution, numerical analysis must be resorted to, although a partial analytic

solution is obtained for one particular form of shear profile. Details of two numerical solutions are given and it is shown how the solution may be improved so as to remove certain singularities which result from the linearisation and also to permit the formation of shock waves to be included in the analysis. *J. J. Mahony* (Sydney)

3351:

Galin, G. Ya. A theory of shock waves. Dokl. Akad. Nauk SSSR **127** (1959), 55–58 (Russian); translated as Soviet Physics. Dokl. **4** (1960), 757–760.

An extension is given of Weyl's theory of shock waves [Comm. Pure Appl. Math. **2** (1949), 103–122; MR **11**, 626] for fluids or solids without restriction on the equation of state. *J. J. Mahony* (Sydney)

3352:

Johnson, W. R.; LaPorte, Otto. Interaction of cylindrical sound waves with a stationary shock wave. Phys. Fluids **1** (1958), 82–94.

The disturbance field of the flow of a perfect gas with two regions of uniform flow separated by a normal shock wave is considered when a harmonic acoustic line source is present. The sound field of the source is represented as the composition of plane waves whose interaction with the shock wave is determined. Thus the complete field is obtained in terms of complete integrals for which asymptotic expressions are given for large values of a non-dimensional frequency parameter based essentially on distance from the source and the speed of sound.
J. J. Mahony (Sydney)

3353:

Taniuti, Tosiya. On the wave propagation in the non-linear fields. Progr. Theoret. Phys. **17** (1957), 461–486.

In this paper the wave equation proposed by Born and employing the Lagrangian

$$L = l^2\{1+l^{-2}(\varphi_x^2-\varphi_t^2)\}^{1/2}$$

is discussed. The associated nonlinear wave equation is made into a first order system and is put into normal form by introducing the two Riemann invariants as new variables. It turns out that the characteristic speed with which each Riemann invariant propagates is a function of the other invariant only. This "linear degeneration" has the consequence that waves don't break, i.e., no shocks arise in waves of finite amplitude.

A similar examination of a one-dimensional wave equation arising in relativistic hydrodynamics, derived from the Lagrangian $L=\frac{1}{2}(\varphi_t\cdot\varphi_t)^2$, shows that here the situation is different, i.e., that shock waves are created.

There is a discussion of the asymptotic distribution of energy in waves governed by both types of equations.
P. D. Lax (New York)

3354:

Taniuti, Toshiya. On the wave propagation in the non-linear fields. II. Progr. Theoret. Phys. **20** (1958), 529–541.

In this paper the author studies wave equations derived from Lagrangians of the form $L=L(Q,\varphi)$, $\varphi=\frac{1}{2}\varphi_\nu\varphi_\mu$. When L is a linear function of Q the associated Euler equation is semilinear; the propagation of discontinuities in higher derivatives of solutions of such equations is investigated. In the quasilinear case solutions are studied by the hodograph method. The propagation of finite disturbances for large t is investigated, without taking into account the formation and decay of shock waves.

Finally the author considers quasilinear equations depending on a parameter ε which become linear for $\varepsilon=0$. The character of the solutions near $\varepsilon=0$ is studied.
P. D. Lax (New York)

3355:

Sichel, Martin. Higher-order corrections to Taylor's solution for weak, normal, shock waves. J. Aerospace Sci. **27** (1960), 635–636.

3356:

Gundersen, Roy. The piston-driven shock. J. Aero/Space Sci. **27** (1960), 467–468.

3357:

Duff, Russell E. Relaxation time for reactions behind shock waves and shock wave profiles. Phys. Fluids **1** (1958), 242–245.

3358:

Fay, James A. Two-dimensional gaseous detonations: velocity deficit. Phys. Fluids **2** (1959), 283–289.

Author's summary: "The measured velocity of gaseous detonation waves is less than that predicted by the Chapman-Jouguet plane wave theory. The velocity deficit (difference between theoretical and measured velocities) has been found earlier to vary inversely with the tube diameter and initial pressure. A quantitative explanation of this effect is advanced by determining the growth of the viscous boundary layer on the tube wall and its effect upon the flow in the reaction zone of the detonation front. It is proposed that the boundary layer displacement effect within the reaction zone produces a uniform flow divergence throughout the detonation front. The velocity deficit due to this two-dimensional flow is determined, using measured values of reaction zone thickness. The agreement of the velocity deficit with measured values is within a factor of two for the five hydrogen-oxygen-inert gas mixtures and one acetylene-oxygen mixture for which sufficient data are available."

3359:

Clarke, J. F. The linearized flow of a dissociating gas. J. Fluid Mech. **7** (1960), 577–595.

The equations governing the steady two-dimensional flow of an ideal symmetrical diatomic dissociating gas are linearised on the assumption that departures from equilibrium values are small. The equations take a useful form if the equilibrium atom mass fraction is regarded as a function of the local pressure and entropy values.

The further assumption is made that disturbances to a uniform free stream are small, and the equations applied to supersonic flow past a sharp corner such that the deflection is small. In contradistinction to the Prandtl-Meyer flow for an inert gas, the pressure and temperature increase steadily on the downstream side of the corner. These

effects are due respectively to the presence of vorticity because of non-equilibrium flow and the liberation of dissociation energy because of the fall of atom concentration. The solution obtained compares reasonably with those obtained by the method of characteristics.
H. C. Levey (Perth)

3360:

Grasyuk, D. S. Scattering of sound waves by the uneven surface of an elastic body. Akust. Ž. **6** (1960), 30–33 (Russian); translated as Soviet Physics. Acoust. **6**, 26–29.

Author's summary: "The problem of the scattering of sound by an uneven boundary between a fluid and solid object is solved. The unevenness is assumed to be periodic with an amplitude small in comparison with the wavelength of the incident radiation. Applying the approximate method of Rayleigh, the amplitudes of the waves sliding along the surface and of the shear waves in the solid object are determined. It is shown that with small angles of incidence (below 12°) the shear waves can be neglected."

3361:

Lapin, A. D. Sound propagation in a waveguide having rectangular grooves in the walls. Akust. Ž. **6** (1960), 237–243 (Russian); translated as Soviet Physics. Acoust. **6**, 233–238.

3362:

Tartakovskiĭ, B. D. Sound field in the focal plane of spherically converging beams. Akust. Ž. **6** (1960), 96–100 (Russian); translated as Soviet Physics. Acoust. **6**, 92–96.

3363:

Lysanov, Yu. P. The problem of surface resonance on a sinusoidal surface. Akust. Ž. **6** (1960), 77–80 (Russian); translated as Soviet Physics. Acoust. **6**, 71–74.

Author's summary: "Based on a modified method of Rayleigh, the effect of surface resonance on an inclined sinusoidal surface with a small amplitude is investigated."

3364:

Kryažev, F. I. Sound field of a first-order plate wave in a layer of water. Akust. Ž. **6** (1960), 65–76 (Russian); translated as Soviet Physics. Acoust. **6**, 60–70.

Author's summary: "The field of a first-order plate wave in a plane-parallel water layer lying over a bottom having no shear elasticity is considered. The results of experiments devoted to a study of the properties of a first-order plate wave with the propagation of sound in shallow water under actual conditions and a determination of the properties of the bottom on the basis of the plate-wave characteristics are given. A comparison of the theoretical and experimental data is made."

3365:

Kaspar'yanc, A. A. Nonstationary radiation of sound by a piston. Akust. Ž. **6** (1960), 52–56 (Russian); translated as Soviet Physics. Acoust. **6**, 47–51.

Author's summary: "The nonstationary field of short-wavelength sound waves formed by a plane piston radiator, the action of which is begun in a gas initially at rest, is studied. In the process of building up the stationary regime a motion having the nature of a pulse, the form of which is closely connected with the form of the radiator, is observed, and the intensity depends on the phase of the radiator action at the instant it is begun."

3366:

Morrow, Charles T. Random vibration. J. Acoust. Soc. Amer. **32** (1960), 742–748.

Author's summary: "There is, at present, a great interest in random vibration as a special aspect of shock and vibration, especially in relation to effects on equipment. An attempt will be made to explain what is meant by random vibration as opposed to other excitations that might be confused with it, what is involved in random vibration testing, what the corresponding objectives are, why there is controversy about these general subjects, and how some of the controversies may be resolved."

3367:

Lyamšev, L. M. Theory of sound radiation by thin elastic shells and plates. Akust. Ž. **5** (1959), 420–427 (Russian); translated as Soviet Physics. Acoust. **5** (1960), 431–438.

The radiation of sound by a thin elastic shell oscillating under external forces distributed in either a regular or random manner over its surface is considered. The method consists in obtaining an integral relation between the required solution of the radiation problem and the known solution of the problem of diffraction of the field of a point source in the space outside the shell. The solution of the former problem is then expressible in terms of the latter by a quadrature. The method is used to obtain results for a cylindrical shell, a spherical shell, and a thin plate, each vibrating under the action of randomly distributed forces.
R. N. Goss (San Diego, Calif.)

3368:

Kontorovič, V. M. Reflection and refraction of sound by shock waves. Akust. Ž. **5** (1959), 314–323 (Russian); translated as Soviet Physics. Acoust. **5** (1960), 320–330.

From the author's summary: "The reflection and refraction of small perturbations, mainly sound, of surfaces of discontinuity in a liquid or gas are considered. The Rankine-Hugoniot conditions are assumed to apply to the discontinuity. The coefficients of reflection and transmission are determined and geometric laws of reflection and refraction given." *J. J. Mahony* (Sydney)

3369:

Acrivos, A.; Shah, M. J.; Petersen, E. E. On the flow of a non-Newtonian liquid on a rotating disk. J. Appl. Phys. **31** (1960), 963–968.

The authors postulate an empirical power law relationship between horizontal shear stress and the vertical velocity gradient for a non-Newtonian fluid in motion on a rotating horizontal disk. The resulting equations are solved for various kinds of flow. It is not clear what form

the corresponding equations would take for a general kind of motion, or whether such equations could be formulated to obey the necessary invariance principles.

J. E. Adkins (Providence, R.I.)

3370:

Stuetzer, Otmar M. Instability of certain electrohydrodynamic systems. Phys. Fluids **2** (1959), 642–648.

Author's summary: "The production of ion drag pressure under dynamic conditions, i.e., with the carrier medium in motion, is theoretically investigated. It is shown that for constant applied voltage the pressure increases with increasing velocity of the carrier fluid. This can lead to instability of the system which is theoretically discussed and experimentally demonstrated."

3371:

Kraus, L.; Yoshihara, H. Electrogasdynamic motion of a charged body in a plasma. J. Aero/Space Sci. **27** (1960), 229–233.

After setting the problem of the motion of a body through an ionized gas (plasma) in context with related problems of unionized gas flow, the authors briefly review the bases for treating the plasma flow problem by the ordinary equations of gas dynamics. They point out that in the limit of small mean free path for interparticle collisions, the first few velocity moments of the Boltzmann equation yield the usual equations of gas dynamics with a scalar pressure. They assert that even in the opposite limit of a charged body moving through the very tenuous plasma of interplanetary space, the moment equations can be reduced to a form analogous to those for the linearized continuous flow by introducing a macroscopic body force resulting from the collective electrical effects and a corresponding directed particle velocity. An approximate linearization is to be carried out "based on the difference between the directed and random velocity".

With this introduction, they offer their basic equations which are the steady state momentum flow equation, conservation of particles and adiabatic equation of state for ions, Boltzmann isothermal density distribution for electrons in a potential and Poisson's equation for the potential. A linearization of this set is carried out in terms of small perturbations from equilibrium values of the dependent variables. The resulting coupled, second order, linear, homogeneous partial differential equations are analyzed qualitatively in terms of characteristics. "If the free stream velocity V_0 is less than the ion sound speed, all roots are imaginary, and the [combined fourth order] equation is completely elliptic. If V_0 is greater than the ion sound speed, we have two real and two imaginary roots. That is, the equation is two fold elliptic and two fold hyperbolic. The latter case is of particular interest, since here one expects many of the effects of supersonic aerodynamics." A brief discussion of boundary conditions is given.

An example which allegedly allows an approximation which reduces the combined fourth order differential equation to one of second order is worked out. For this case the wave and electrostatic drags are computed. However, the total charge on the body is taken as zero, contrary to the negative charge to be expected on physical grounds as argued in their introduction.

J. E. Drummond (Seattle, Wash.)

3372:

Chang, C. C.; Yen, J. T. Rayleigh's problem in magnetohydrodynamics. Phys. Fluids **2** (1959), 393–403.

The authors consider the impulsive motion of an infinitely extended viscous incompressible imperfect electrically conductive fluid, started from rest by an electrically perfect conductive infinite flat plate moving with constant velocity parallel to its plane; the plate is acted upon by a transverse uniform magnetic field fixed in direction. Both the induced electric and magnetic fields are treated in the analysis, which is applied in the (y, t) plane. The basic equations are: (1) the continuity equation; (2) the equations of motion; (3) Maxwell's equations; (4) Ohm's law; (5) the conservation of current. These lead to a system of fourth order equations in four unknowns plus boundary and initial data (and also four algebraic equations and an equation for pressure when a linearization is assumed). The system of linear differential equations is solved by the Laplace transform. The exact solution is obtained and analyzed when χ, the ratio of the magnetic permeability to the kinematic viscosity, is one. In some other cases ($\chi \to \infty$, $20 < \chi < 400$, $(1/400) < \chi < (1/20)$), the approximate solution is discussed. Profiles of the electric, magnetic fields, body force, coefficient of skin friction are calculated for several values of χ. Finally, two tables of physical constants, used in the paper, for four fluids are given.

N. Coburn (Ann Arbor, Mich.)

3373:

Leigh, D. C.; Sutton, G. W. An analogy between magnetohydrodynamics and heat transfer. J. Aero/Space Sci. **27** (1960), 469–470.

3374:

Rand, S. Wake of a satellite traversing the ionosphere. Phys. Fluids **3** (1960), 265–273.

Author's summary: "The particle treatment is applied to a study of the structure of the wake behind a charged body moving supersonically through a low-density plasma. For the case of a body whose dimensions are considerably smaller than a Debye length, a solution is obtained which is very similar in structure to the solution obtained by using the linearized fluid dynamics equation. For the case of a disk whose radial dimensions are much larger than a Debye length, two conical regions are found in the wake. At the surface of each of these cones, over thicknesses of the order of a Debye length, the ion and electron densities are increased over their ambient values. Formulae for the electrohydrodynamic drag on a wire, and on a large disk are obtained."

3375:

Ladyženskaya, O. A.; Solonnikov, V. A. On the solvability of unsteady motion problems in magnetohydrodynamics. Dokl. Akad. Nauk SSSR **124** (1959), 26–28. (Russian)

3376:

Wright, J. K.; Black, M. C. A theory of electromagnetically driven shock waves. J. Fluid Mech. **6** (1959), 289–301.

Les auteurs étudient le problème de la génération des ondes de choc par des forces électromagnétiques. Le phénomène est supposé unidimensionnel; une surface de courant sur laquelle la pression du fluide équilibre la "pression magnétique", joue le rôle d'un piston, dont le mouvement engendre un choc. Au voisinage de l'instant initiel des "solutions de similitude" sont obtenues; des développements de Taylor et des calculs numériques sont utilisés ensuite. *H. Cabannes* (Paris)

3377:

Citron, Stephen J. Steady-state melting of a semi-infinite medium with temperature-dependent properties. J. Aero/Space Sci. **27** (1960), 470–472.

3378:

Kemp, Nelson H.; Petschek, Harry E. Theory of the flow in the magnetic annular shock tube. Phys. Fluids **2** (1959), 599–608.

Un champ magnétique peut être utilisé, pour propager une onde de choc dans un tube à choc annulaire. La théorie d'un tel tube à choc est traitée; des calculs détaillés relatifs aux diverses grandeurs qui caractérisent l'état du gaz sont effectués. *H. Cabannes* (Paris)

3379:

Lyubarskiĭ, G. Ya.; Polovin, R. V. The piston problem in magnetic hydrodynamics. Dokl. Akad. Nauk SSSR **128** (1959), 684–687 (Russian); translated as Soviet Physics. Dokl. **4** (1960), 977–980.

The authors consider the motion of a perfectly conducting piston into a perfectly conducting fluid. An Alfvén wave discontinuity does not occur during the motion of the piston. For compressive motion two magnetohydrodynamic shock waves are formed (fast and slow); two simple waves occur in a rarefaction.

H. P. Greenspan (Cambridge, Mass.)

3380:

Berezin, O. A. On a particular solution of equations of magnetic gas dynamics. Vestnik Leningrad. Univ. **15** (1960), no. 1, 107–110. (Russian. English summary)

Author's summary: "The particular solution of the equations of magnetic gas dynamics depending on one arbitrary function is given. The solution has been conjugated with the shock wave propagating in motionless gas in the medium with some initial density $\rho_1(x_2)$, pressure $P_1(x_2)$ and strength of the magnetic field $h_1=h_1^0-P_1(x_2)$. The form of the functions $\rho(x_2)$, $P_1(x_2)$ and the shock wave motion law are determined from the conditions of the dynamical compatibility."

3381:

Žimartbaev, M. T. Absorption of sound and the width of shock waves in relativistic hydrodynamics. Ž. Èksper. Teoret. Fiz. **37** (1959), 1000–1004 (Russian); translated as Soviet Physics. JETP **10** (1960), 711–713.

The dissipation of sound energy is calculated from equations giving the rate of increase of entropy in terms of the viscosity and heat conductivity of a relativistic fluid. No account is taken of other effects which transfer the spatial organisation of a wave into other organized energy forms. (An example is Landau damping which transfers the organization to velocity space. This might be especially important for a relativistic fluid.) Drawing many equations from statistical physics and thermodynamics (with some confusion of symbols) the author outlines also the derivation of shock wave thickness to second order in wave amplitude.

J. E. Drummond (Seattle, Wash.)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 3168, 3169, 3181, 3371, 3586.

3382:

Wynne, C. G. Lens designing by electronic digital computer. I. Proc. Phys. Soc. **73** (1959), 777–787.

The numerical methods currently used in automatic lens design programs are carefully discussed and reasons are advanced for the poor convergence often experienced in practice when finite differences are used to approximate derivatives. The difficulties encountered in using either the Newton-Ralphson method or relaxation methods are generally well known and so are only briefly described, but the author devotes more attention to a very illuminating discussion of the method of steepest descent as applied to automatic lens design.

Finally, a description is given for an iterative method called SLAMS (Successive Linear Approximations with Maximum Steps). If A_i are the aberration functions of a lens system with design parameters x_i, then SLAMS minimizes $\phi=\sum A_i^2$ subject to the restriction that $\sum x_i^2$ is less than some prescribed bound. It is shown that ϕ decreases with successive iterations.

G. L. Walker (Providence, R.I.)

3383:

Nunn, M.; Wynne, C. G. Lens designing by electronic digital computer. II. Proc. Phys. Soc. **74** (1959), 316–329.

A computing program is described for the automatic lens design method SLAMS of Wynne [see preceding review]. The results of applying the method to three design problems are quoted.

G. L. Walker (Providence, R.I.)

3384:

Toman, K. Focusing, defocusing, and refraction in a circularly stratified atmosphere. J. Res. Nat. Bur. Standards Sect. D **64D** (1960), 287–288.

This paper only concerns very general properties of ray trajectories in spherically symmetric media. Discussed are the focusing and defocusing phenomena in such a medium (even existing for a narrow beam propagating perpendicularly to the stratification), the astronomical refraction, and the path lengths of rays. Much attention is given to the symmetry between rays leaving a point source along directions pointing upwards and downwards with respect to the level of stratification. No details of the computation method are given.

H. Bremmer (Eindhoven)

3385:

Picht, Johannes. Überlegungen zur Theorie der Reflexion an einem mit konstanter Winkelgeschwindigkeit rotierenden Spiegel. Z. Physik **156** (1959), 468–487.

The author writes a preliminary paper dealing with the reflection of an arbitrary wave by a moving mirror, which has either a translatory movement of constant velocity or rotates with constant angular velocity.

The author finds that in the latter case the rotating plane mirror has the same effect as a concave cylindrical mirror. Relativistic considerations are at least partially taken account of. The paper suggests at the end further theoretical and experimental investigations of the problem.

M. Herzberger (Rochester, N.Y.)

3386:

Cheng, David K. Relations concerning refracting surfaces, wavefronts, and phase errors. J. Franklin Inst. **269** (1960), 184–195.

The author develops a formula to calculate the characteristic function of a homogeneous dielectrical lens, and therefore the phase relationship existing between different rays coming to a focus. This if carried out in detail would allow one to compute the wave surface of the emerging bundle. The relations are given in a form which permits one to trace each single ray through a lens; the results for tracing the manifold of all rays are not given in an explicit form. *M. Herzberger* (Rochester, N.Y.)

3387:

Blottiau, Félicien; Bertrand, Gérard. Répercussion des erreurs spectrophotométriques sur les calculs colorimétriques. Rev. Opt. **39** (1960), 209–223. (English summary)

Authors' summary: "A general purpose elementary procedure is developed to determine, with the aid of the standard deviations observed in the various ordinates of a spectral curve, the symmetry centered $\mathscr{S}$ surface having two singular points, which bounds the uncertainty volume in a tridimensional representation of colours. This procedure is also used to obtain the $\mathscr{C}$ curve that bounds the uncertainty area in the chromaticity diagram, either indirectly as derived from the $\mathscr{S}$ surface, or, which is more rapid, directly through a calculation of uncertainties only in the chromaticity coordinates x, y. A practical application of this method is thoroughly discussed on an example; it shows that only a rather rough estimate would allow the $\mathscr{C}$ curve to be assimilated with an ellipse."

3388:

Gutšabaš, S. D. Light dispersion in the medium near the reflecting surface. Vestnik Leningrad. Univ. **15** (1960), no. 1, 152–158. (Russian. English summary)

Author's summary: "The problem of light transfer in the plane-parallel strata limited by a reflecting surface is considered. The resolvent of the integral equation in this case is shown to be expressed by the resolvent integral equation without reflecting surface. The problem is solved by the probability method. Two particular cases are considered as examples: (1) the medium is illuminated by parallel rays; (2) the sources of illumination are distributed in the medium uniformly."

3389:

Caldirola, Piero. Formulazione Lagrangiana e Hamiltoniana del moto classico dell'elettrone irraggiante. Ist. Lombard Accad. Sci. Lett. Rend. A **93** (1959), 439–445.

In this article the author gives a procedure of transforming the equation of motion of a non-radiating electron in a potential field to the Lagrangian and Hamiltonian formalism, where the dissipative forces are expressed by the first and second derivatives of the acceleration of the electron. Here two cases are treated: (a) the first two derivatives of the acceleration appear in the equations of motion; (b) higher order derivatives are included. By taking a Lagrangian function quadratic in the acceleration with a coefficient (factor) depending on the time t, the author has derived the equations of case (a) by determining the coefficient of the quadratic form, whose dependence on t is a linear exponential decay curve. For the case (b), the Lagrangian contains in addition a quadratic form in the second derivative of the acceleration with a constant factor. The equations of motion reduce to the case (a) if this coefficient is given an appropriate value and higher derivatives in the acceleration are neglected. The Hamiltonian function corresponding to the Lagrangian of case (a) is obtained by introducing canonical variables $q_i = x_i$, $Q_i = \dot{x}_i$. The corresponding moments p_i, P_i are found from the Lagrangian in the usual way. This study forms the starting point from which the quantum mechanical treatment of this problem could be pursued.

N. Chako (Flushing, N.Y.)

3390:

Yadavalli, S. V. Non-linear behaviour of a modulated electron beam in the presence of a velocity distribution. J. Electronics Control (1) **8** (1960), 365–375.

Author's summary: "A procedure based on the Boltzmann equation is given for evaluating the harmonic currents in an electron beam in the presence of a velocity distribution. Employing the above method, the second harmonic current in a drifting (initially velocity modulated) beam possessing a velocity distribution is evaluated."

3391:

Caianiello, E. R.; Gatti, G. Proprietà focalizzanti del campo di Biot-Savart. Nuovo Cimento (10) **12** (1959), 469–476. (English summary)

The purpose of this paper is to study the focusing properties of high magnetic (axial) fields produced by high intensity cylindrical currents for charged particles having a finite initial angular velocity. The intensity of the current is assumed to vary slowly in comparison to the period of rotation of the electron.

The results of calculation show that the motion of the electron is stable in the radial and angular coordinates (r and θ vary between finite limits) whereas the variation of the axial coordinate is proportional to the time. This leads to the practical possibility of designing a particle accelerator. *N. Chako* (Flushing, N.Y.)

3392:

Seymour, P. W. Drift of a charged particle in magnetic field of constant gradient. Austral. J. Phys. **12** (1959), 309–314.

An exact solution in terms of elliptic integrals is obtained for the drift velocity of a charged particle moving in magnetic field of constant gradient, $B_z = \lambda x$. An approximation leads to Alfvén's first order result [Ark. Mat. Astr. Fyz. **27A** (1941), no. 22; MR **6**, 168] and to the case of the circular orbit. The solution includes the case of zero mean field, for which perturbation methods are inappropriate. *J. E. Rosenthal* (Passaic, N.J.)

3393:

Tonks, Lewi. Self-consistent field and motion of electrons which have a range in canonical angular momentum in a uniform magnetic field. Phys. Rev. (2) **118** (1960), 390–398.

Author's summary: "The self-consistent theory of relativistic electrons circulating in a uniform impressed magnetic field, the 'Astron problem,' has been generalized to the extent that a range of canonical angular momentum among monoenergetic electrons has been treated. For simplicity, the density distribution in phase space has been chosen to be uniform over a finite momentum range. Just as in the single-momentum case, field reversal is found, but new field and spatial density configurations appear. The uniform distribution is found to be consistent with isotropic regions of constant spatial density and constant magnetic field. The thickness of transition layer between vacuum and such a region conforms, within limits, to an empirical relation previously found. The limit to the number of electrons per unit axial length of layer still exists. The curves relating the ratio of internal to external field to the layer strength still show multiple values of both ratio and strength in certain ranges. Trajectories have been calculated and plotted for several cases."

3394:

Fano, U. Normal modes of a lattice of oscillators with many resonances and dipolar coupling. Phys. Rev. (2) **118** (1960), 451–455.

Author's summary: "The normal modes of a lattice of coupled dipoles are studied as a model of the collective excitations of electrons in condensed materials. Two types of oscillations are found in which electrostatic coupling has a dominant influence. One of them is analogous to the oscillation of an electron plasma and has a high dipole moment. Other collective oscillations have a low net dipole moment, owing to destructive interference between out-of-phase components. These two types of oscillation occur in systems with a sufficiently high density of oscillator strength in space and in spectrum. A simple estimate indicates that most condensed materials fulfill this condition."

3395:

Romig, Mary F. Steady state solutions of the radiofrequency discharge with flow. Phys. Fluids **3** (1960), 129–133.

Author's summary: "The electron density distribution and diffusion length have been investigated for a steady-state, diffusion-controlled radiofrequency discharge acting over a finite portion of an infinite cylinder in which there is a uniform axial gas flow. This model simulates to some extent the flow in a plasma wind tunnel. A qualitative relationship is obtained for the influence of active cylinder length and gas velocity parameter on the diffusion length. The effect of these parameters on the electric field necessary to sustain the discharge is also discussed. It is shown that the peak of the electron density distribution shifts downstream with increasing gas velocity, but never leaves the region of production. A numerical example is calculated for the case of helium, indicating that while there is moderate effect on breakdown parameters, the ambipolar case may be changed considerably by the presence of flow."

3396:

Dawson, John M. Plasma oscillations of a large number of electron beams. Phys. Rev. (2) **118** (1960), 381–389.

Author's summary: "Longitudinal oscillations of a large number of electron beams are investigated. The normal modes for the beams are found. An orthogonality relation between the modes is obtained and is used to solve the initial value problem and the problem of forced oscillations. It is demonstrated that no signal propagates faster than the fastest beam. The problem of passing to the limit of a continuous velocity distribution is considered in detail. It is shown that in the limit the results of Landau, Van Kampen, and others are recovered. The problem of Landau damping is discussed from the point of view of the beams."

3397:

Sturrock, P. A. Action-transfer and frequency-shift relations in the nonlinear theory of waves and oscillations. Ann. Physics **9** (1960), 422–434.

Certain relations which have arisen in the theories of electrical networks, electron tubes, plasma oscillations, and particle accelerators are shown to be special cases of general relations attributable to any system which may be described by a Hamiltonian. If such a system is analyzed into an interacting set of modes (waves or oscillators), it is found that the rates of transfer of action to or from members of a group of interacting modes are in integral ratios, the integers being determined by the interaction, and are related in magnitude to the energy associated with the interaction. These relations lead to the Manley Rowe expressions if an assumption of weak coupling is made. Furthermore, a conjugate set of relations is found that leads under the assumption of weak coupling to the following statement: the fractional shifts in frequency of members of a group of interacting modes, when multiplied by the energies in those modes, are in integral ratios, the integers being determined by the interaction, and are related in magnitude to the energy associated with the interaction. The "frequency-shift relations" provide useful information relating the frequencies of uncoupled modes and the frequencies of excitation of these modes to the partition of energy of this excitation among these modes. This application is demonstrated by a simple example.

H. A. Haus (Cambridge, Mass.)

3398:

Biermann, L.; Pfirsch, D. Kooperative Phänomene und Diffusion eines Plasmas quer zu einem Magnetfeld. I. Z. Naturforschg. **15a** (1960), 10–12.

Auch in der Astrophysik und auch in der Theorie der Kernfusionsreaktoren ist es ein wichtiges Problem, wie schnell ein durch ein Magnetfeld zusammengehaltenes Plasma auseinanderfliesst. Aus dem Ohmschen Gesetz und dem Induktionsgesetz folgt im einfachsten Falle

$$(1)\quad -\frac{1}{c}\dot{\mathfrak{B}} \equiv -\frac{1}{c}(\dot{\mathfrak{B}} - \mathrm{rot}\,[\mathfrak{v}\times\mathfrak{B}]) = \mathrm{rot}\,(\eta\mathfrak{j}) - \mathrm{rot}\,\mathfrak{E}^e,$$

wo $\mathfrak{v}$ die makroskopische Geschwindigkeit, η den elektrischen Widerstand, $\mathfrak{j}$ die Stromdichte und $\mathfrak{E}^e$ die aus Druckgradienten (und evtl. Beschleunigungen) resultierenden elektromotorischen Kräfte bedeutet. Mit Hilfe der ersten Maxwellschen Gleichung erhält man dann für die Zeitskala, in der das Plasma aus dem Querschnitt πs^2 herausdiffundiert

$$(2)\quad s^2/t \approx c^2\eta\beta,$$

wo β das Verhältnis des Gasdruckes zu $(1/8\pi)B^2$ ist. Die Erfahrung zeigt jedoch, dass das Plasma viel schneller zur Wand diffundiert, und die theoretische Untersuchung dieses Widerspruches ist das Ziel der vorliegenden Arbeit. Erstens wird gezeigt, dass wenn man die Diffusion der Ladungsträger als einen stochastischen Prozess auffasst, dann aus dieser Annahme (2) bis auf einen logarithmischen Faktor hergeleitet werden kann. Die erwähnte Erfahrung muss also in überthermischen Schwankungen ihre Ursache haben. Es wird darauf aufmerksam gemacht, dass in einem Druckgradienten, der nicht allein durch ein Potentialfeld aufrechterhalten wird, lokal keine Maxwellsche Geschwindigkeitsverteilung besteht. Mit Hilfe einer stark schematisierten Rechnung wird dann gezeigt, dass in diesem Falle tatsächlich viel grössere Energieschwankungen entstehen, welche grössenordnungsmässig mit den von D. Gábor, E. A. Ash und D. Dracott [Nature (London) **176** (1955), 916–919] auf experimentellem Wege gefundenen übereinstimmen. *T. Neugebauer* (Budapest)

3399:

Pfirsch, D.; Biermann, L. Kooperative Phänomene und Diffusion eines Plasmas quer zu einem Magnetfeld. II. Z. Naturforschg. **15a** (1960), 14–18.

Als Fortsetzung des ersten Teiles dieser Arbeit [siehe vorangehende Besprechung] wird besonders an zwei Beispielen (welche in inhomogenen Plasmen auftreten können), die Frage untersucht, ob in diesen Plasmen mikroskopische Instabilitäten möglich sind. Den Ausgangspunkt der Rechnungen bilden die Einteilchen-Boltzmann-Gleichung und die Maxwellschen Gleichungen. Zuerst wird der Fall betrachtet, in dem ein Dichtegradient mit Teilchendiffusion jedoch ohne Magnetfeld vorhanden ist. Ausgegangen wird von der Verteilungsfunktion

$$(1)\quad f(x, \mathfrak{v}) = n[x - v_x\tau(v)]f_0(v),$$

wo f_0 die Maxwell-Verteilung und $n(x)$ die Teilchendichte bedeutet. $\tau(v)$ hängt von der freien Weglänge ab (und ist eine Art Flugzeit). (1) löst genähert die Planck-Fokkersche Gleichung. Weitere Rechnungen zeigen, dass wenn man von der linearisierten Theorie ausgeht, dann in diesem Falle Instabilitäten nicht entstehen und deshalb ist es wahrscheinlich, dass die in I erwähnten Instabilitäten nur aus nichtlinearen Theorien folgen.

Als zweites Beispiel wird ein Plasma mit Dichtegradient im Magnetfeld jedoch ohne Diffusion betrachtet. Die Verteilungsfunktion wird in diesem Fall ebenfalls explizit angegeben und aus den weiteren Betrachtungen folgt, dass jetzt tatsächlich solche Instabilitäten auftreten können, welche aus der makroskopischen Theorie nicht folgen. Zuletzt wird der Fall des Dichtegradienten mit Teilchendiffusion und gleichzeitigem Magnetfeld ganz kurz besprochen und im Anhang wird die linearisierte Boltzmann-Gleichung nach dem Charakteristikverfahren gelöst. *T. Neugebauer* (Budapest)

3400:

Kruskal, M. D.; Kulsrud, R. M. Equilibrium of a magnetically confined plasma in a toroid. Phys. Fluids **1** (1958), 265–274.

The authors discuss the properties of a magnetically confined plasma in equilibrium in a toroid whose surface is a surface of constant pressure and is made up of lines of magnetic force. A number of quantities are defined on these magnetic surfaces and relations found between them. A variational principle of these equilibrium configurations is derived. One of the consequences is that it provides a characterization of equilibria by the values of certain invariants connected with the magnetic surfaces. Finally a complete set of equations governing a steady state plasma slowly diffusing through the magnetic field out of the toroidal tube is obtained. It is shown that the system of equations has a solution in the limiting case of low pressure. *F. C. Auluck* (Delhi)

3401:

Kruskal, M. D.; Oberman, C. R. On the stability of plasma in static equilibrium. Phys. Fluids **1** (1958), 275–280.

In this paper criteria useful in the discussion of stability of plasmas in static equilibrium are derived from the Boltzmann equation in the small m/e limit. These criteria are obtained from the variation of the energy due to a perturbation, subject to the constraint that all regular, time-independent constants of the motion have their equilibrium values. The first-order variation vanishes and the second-order variation yields a quadratic form in the displacement parameter. The positive-definiteness of this form is a sufficient condition for stability. Several theorems are given comparing stability under the present theory with that under conventional hydromagnetic fluid theories. The method is also capable of generalization to systems in which the effect of collisions is included. *F. C. Auluck* (Delhi)

3402:

R.-Shersby-Harvie, R. B. Radio-frequency forced oscillations in non-uniform plasmas. J. Electronics Control (1) 8 (1960), 421–430.

Author's summary: "A wave equation and associated equations is derived for periodic forced oscillations in a collision free plasma. This set of equations is rather complicated and small signal approximations are derived, which are applicable to plasma confinement problems. One approximation is applied to the case of a fast E_θ

travelling wave in plasma and its behaviour is shown to be markedly different from the well-known case of an E_0-wave at cut-off. Some difficulties with small signal theories are indicated and the need for further clarification pointed out."

3403:

Cerkovnikov, Yu. A. The question of convectional instability of a plasma. Dokl. Akad. Nauk SSSR **130** (1960), 295–298 (Russian); translated as Soviet Physics. Dokl. **5**, 87–90.

3404:

Cushing, Vincent; Sodha, Mahendra S. Confinement of plasma by standing electromagnetic wave. Phys. Fluids **2** (1959), 494–498.

Authors' summary: "In this paper the authors have discussed the confinement of plasma by a one-dimensional stationary electromagnetic wave. Their analysis is similar to that carried out by former workers for a $TM_{0,1}$ mode but they have examined the assumptions limiting the applicability of present theories in detail. One of the conditions for the applicability of the present theory is that the frequency of the electromagnetic wave should be of the same order or greater than the plasma frequency at the nodes where particle density is maximum. This condition makes the present theories nonapplicable in cases of thermonuclear interest."

3405:

Šafranov, V. D. Equilibrium of a plasma toroid in a magnetic field. Ž. Èksper. Teoret. Fiz. **37** (1959), 1088–1095 (Russian); translated as Soviet Physics. JETP **10** (1960), 775–779.

After stating the basic partial differential equations connecting Ψ (related to the ϕ component of the vector potential of the magnetic field in an axially symmetric plasma in dynamic equilibrium) with the current densities and the (assumed scalar) plasma pressure, the author describes briefly the boundary condition problem. Instead of solving this for the plasma configuration, he assumes a toroidal configuration, and solves for the external magnetic boundary conditions necessary to maintain a plasma in such an equilibrium, though the nature of the equilibrium itself (i.e., stable, unstable or neutral) is not considered. In addition to assuming the pinch with circular cross section, he also assumes that plasma pressure and the square of axial current (i.e., axial current density integrated from axis to a radius r) are proportional to Ψ. A particular solution of the differential equation for Ψ in terms of the proportionality constants is given simply in closed form in spherical coordinates and then expanded in toroidal form so that it, added to the general solution which is obtained in toroidal coordinates, may be compared with the constant pressure condition on the surface of the torus and a finiteness condition on the solution within the torus. The solution satisfying these conditions and allowing for constant azimuthal current density of arbitrary magnitude on the surface of the torus is obtained. It is used to compute the part of Ψ due to current in external conductors. A number of special cases for small torus curvature (i.e., major radius $\gg$ cross sectional radius) are considered. An appendix lists expansions of some of the solution functions.

J. E. Drummond (Seattle, Wash.)

3406:

Rand, S. Electrostatic field about an ion moving slowly in a plasma. Phys. Fluids **2** (1959), 649–652.

Die Theorie des stationären Debye-Hückelschen Feldes um ein Ion in einem Plasma haben schon viele Autoren auf den Fall verallgemeinert, dass sich dieses Ion mit Unterschallgeschwindigkeit bewegt. Der Ausgangspunkt von allen diesen Untersuchungen war jedoch eine linearisierte Theorie. Ziel der vorliegenden Arbeit ist die Verzerrung des Debye-Feldes für den Fall, dass sich das fragliche Ion mit einer Geschwindigkeit bewegt, welche relativ klein zur mittleren thermischen Geschwindigkeit ist, aus der Partikeltheorie zu berechnen. Dabei wird angenommen, dass $eZ\psi \ll kT$ ist (wo ψ das Potentiel bedeutet), dass die Elektronen- und Ionentemperaturen einander gleich sind und ausserdem dass die Gasdichte recht klein ist.

Den Ausgangspunkt bildet die Poissonsche Differentialgleichung

$$\nabla^2\psi = -4\pi eZ\delta(\mathbf{R}) - 4\pi\rho, \tag{1}$$

wo eZ die Ladung des Ions und $\rho = \rho_+ - \rho_-$ die Ladungsdichte im Abschirmungsfeld bedeutet. δ ist die Diracsche Deltafunktion. ρ_+ erhält man im Unendlichen einfach aus der Boltzmannstatistik und daraus wird berechnet, wie viele dieser Ionen einen Punkt $\mathbf{R}$ nach der Streuung im Potentialfelde ψ erreichen. Wegen der grossen thermischen Geschwindigkeit der Elektronen kann man weiter einfach $\rho_- = \rho_0[1 + e\psi/kT]$ setzen. Nach gewissen Vereinfachungen werden die erhaltenen Formeln für ρ_+ und ρ_- in (1) eingesetzt und die entstehende Differentialgleichung wird gelöst. Die erhaltene Lösung enthält zwei Hilfsfunktionen, deren Werte tabellarisch angegeben werden; die eine wird auch graphisch veranschaulicht. Das Resultat liefert eine Asymmetrie der Ladungsverteilung und demzufolge eine auf das sich bewegende Ion wirkende Hemmungskraft, was das wesentlichste Ergebnis der Arbeit ist.

T. Neugebauer (Budapest)

3407:

Fedorčenko, V. D.; Rutkevič, B. N.; Černyĭ, B. M. Motion of an electron in a spatially periodic magnetic field. Ž. Tehn. Fiz. **29** (1959), 1212–1218 (Russian); translated as Soviet Physics. Tech. Phys. **4** (1960), 1112–1117.

Authors' summary: "An analysis is made of the motion of electrons in a magnetic field, which is constant in time but which is modulated weakly in the longitudinal direction. Under certain conditions a 'resonance' relation obtains between the velocity of the electron, the fixed component of the magnetic field, and the period of the spatial modulation; in this case, the magnetic moment of electron is no longer conserved and the energy is divided between the longitudinal and transverse components of the motion."

J. E. Rosenthal (Passaic, N.J.)

3408:

Wright, James P. Diffusion of charged particles across a magnetic field due to neutral particles. Phys. Fluids **3** (1960), 607–610.

Author's summary: "A calculation of the diffusion of

charged particles across a magnetic field arising from the presence of neutral particles is compared with the diffusion arising from charged particles. The ratio of the flux of charged particles, arising from the presence of neutral particles, and the flux arising from charged particles is found to be of the order 10^2 to 10^5. The actual value of the ratio depends on the types of particles, the temperature, the number densities, and the density gradients."

3409:

Hurst, C. A.; Huxley, L. G. H. The distribution of ions formed by attachment of electrons moving in a steady state of motion through a gas. Austral. J. Phys. **13** (1960), 21–26.

Authors' summary: "The distribution of ions formed by attachment of electrons diffusing through a gas is solved exactly, and the results compared with an approximate calculation given earlier by Huxley. The corrections to the approximate results are inside the present experimental error, and so confirm the satisfactory agreement with experiment already obtained."

3410:

Bonč-Bruevič, V. L.; Mironov, A. G. On the theory of electron plasma in a magnetic field. Fiz. Tverd. Tela **2** (1960), 489–498 (Russian); translated as Soviet Physics. Solid State **2**, 454–462.

Authors' summary: "The behavior of a nondegenerate electron plasma in a constant magnetic field is investigated by the method of Green's functions. The spectrum of plasma oscillations is obtained; in particular the limiting wave number of the vibration is determined and the damping constant calculated taking full account of the quantum correction. The external electric field shielding law (in the presence of a magnetic field) is also investigated."

3411:

Ahiezer, I. A.; Polovin, R. V.; Cincadze, N. L. Simple waves in the Chew-Goldberger-Low approximation. Ž. Èksper. Teoret. Fiz. **37** (1959), 756–759 (Russian); translated as Soviet Physics. JETP **10** (1960), 539–541.

Authors' summary: "Simple waves in a plasma with anisotropic pressure are treated in the Chew-Goldberger-Low approximation. It is shown that there exist three types of simple waves. The direction of the variations of the magnetohydrodynamic quantities in these waves is investigated."

3412:

Konyukov, M. V. Nonlinear Langmuir electron oscillations in a plasma. Ž. Èksper. Teoret. Fiz. **37** (1959), 799–801 (Russian); translated as Soviet Physics. JETP **10** (1960), 570–571.

Author's summary: "An exact solution has been obtained for the nonlinear oscillations of the electron density in a plasma at zero electron temperature. The initial conditions necessary for these oscillations are determined."

3413:

Adler, Richard B.; Chu, Lan Jen; Fano, Robert M. ★Electromagnetic energy transmission and radiation. John Wiley & Sons, Inc., New York-London, 1960. xvii + 621 pp. $14.50.

This is a book treating the main elementary subjects of circuit theory and wave propagation, but in much more detail than the conventional textbooks. The following items, e.g., are discussed thoroughly: the representation of field quantities by complex vectors, the geometrical orientation of the field vectors (elliptic polarization), the splitting of energy in electric and magnetic contributions, the connection between circuits and continuous fields, the propagation direction of energy (Poynting vector), the possibility of attenuated waves in a lossless medium (connected with a reactive component of Poynting's vector), the equivalence of wave-propagation characteristics with parameters of transmission lines, the importance of impedance (including surface impedance) defined by the ratio of an electric and the corresponding magnetic field. These subjects are investigated in separate chapters for plane waves in a homogeneous (lossless or dissipative) medium, for plane waves suffering a perpendicular or oblique reflection and refraction at a plane interface, and for the different modes occurring in a wave guide. The importance of expansions, or also successive approximations, with respect to frequency dependence is stressed in many places (e.g., for "quasistatic fields" defined by a linear approximation in the frequency ω). The book also deals with aspects which are considered only casually in other works. As an example we mention the comparison of the directions of propagation and of maximal attenuation for plane waves in a homogeneous medium; in a lossless medium these directions prove to be orthogonal. Chapter IX deals with an elegant reduction of the propagation in the interspace between infinite cylindrical metallic surfaces to two-dimensional static problems when considering the field distribution over the cross-sections; the elementary equations for transmission along a cable hold here if proper definitions are introduced for the voltage (defined as a line integral of the electric field along a curve connecting two surfaces), and for the capacity and inductance per unit length. The general radiation problem of antennas is only considered in the last chapter. The elementary theory of a Hertzian dipole is represented here in a cumbersome fashion, and could have been shortened considerably by using Dirac's delta function.

The many geometrical considerations are well elucidated by clarifying figures, for instance those explaining complicated polarization conditions, and those representing the phase velocity of wave-guide modes as a function of the frequency. A long list of interesting problems, to be worked out by the reader, appears at the end of each chapter. *H. Bremmer* (Eindhoven)

3414:

Sideriades, Lefteri. ★Méthodes topologiques appliquées à l'électronique. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 84, Paris, 1959. x + 136 pp. 3.190 francs.

Reproduction d'une thèse de doctorat ès-sciences. L'auteur applique des méthodes topologiques développées par Th. Vogel [Ann. Télécommun. **6** (1951), 2–10, 182–

190; MR **13**, 462], et inspirées d'une série de mémoires de H. Poincaré sur l'intégration des systèmes d'équations différentielles [*Œuvres de H. Poincaré, t. 1*, nouveau tirage, Gauthier-Villars, Paris, 1951]. On y trouve, d'une part, une étude mathématique des systèmes non linéaires du premier et du deuxième ordre, d'autre part l'application de ces méthodes à l'étude des réseaux électriques actifs, dans lesquels la non-linéarité des caractéristiques des tubes électroniques joue un rôle essentiel. Notons, en particulier, l'existence d'un cycle limite et les phénomènes de synchronisation et d'irréversibilité de l'amorçage des oscillations. On connait, sur ce sujet, les travaux fondamentaux de Van der Pol. L'étude est, à la fois, qualitative et quantitative. L'auteur admet, pour le faisceau de caractéristiques des tubes électroniques, la forme d'une portion de cylindre parabolique. Les exemples traités sont: l'oscillateur à triode, le multivibrateur d'Abraham-Bloch. Des vérifications expérimentales sont reproduites, sous la forme d'oscillogrammes.

P. M. Poincelot (Issy-les-Moulineaux)

3415:

Wait, James R. Some solutions for electromagnetic problems involving spheroidal, spherical, and cylindrical bodies. J. Res. Nat. Bur. Standards Sect. B **64B** (1960), 15–32.

Solutions are presented for the low-frequency electromagnetic response to an oscillating magnetic dipole by a conducting body of simple shape: prolate spheroid, sphere, and infinite circular cylinder. The spheroid is taken to be a perfect conductor; the sphere and the cylinder are assigned finite conductivity and magnetic permeability. In the author's quasi-stationary approximation, solutions of the wave equation inside the bodies are matched to Laplace's equation solutions outside. Numerical results are shown for some simple locations of dipole source and observed with respect to the body. These are discussed from the point of view of geophysical prospecting.

C. J. Bouwkamp (Eindhoven)

3416:

Bergmann, Otto. Versuch einer endlichen Theorie des Elektrons. Acta Phys. Austriaca **13**, 33–47 (1960).

It is shown that in the classical relativistic theory of charged particles interacting with the Maxwell field, terms describing the self fields of the particles may be consistently dropped (in the mathematical sense) to yield a finite theory. All field reaction effects are, of course, then missing from the resulting theory.

A. Klein (Philadelphia, Pa.)

3417:

Kvasnica, J. A remark on Bopp-Podolsky electrodynamics. Czechoslovak J. Phys. **10** (1960), 81–90. (Russian. English summary)

Author's summary: "A description of the electromagnetic field in vacuum as a specific bi-field, formed by the vectors **E**, **B** and **H**, **D**, leads, on the assumption of non-local relation between the components of the bi-field, to electrodynamics of the type $L(\Box)\,\Box A_\mu = -j_\mu$, where $L(\Box)$ is a rational function of $\Box$. The scattering of electrons in a Bopp-Podolsky field of force is studied and compared with the results of Hoffstadter experiments. A close connection is shown between electrodynamics with higher derivatives and the results and methods of modern quantum electrodynamics (polarization of electron-positron vacuum, Pauli-Villars regularization)."

3418:

Chambers, Ll. G. The electrostatic energy of a two-dimensional system. Proc. Edinburgh Math. Soc. **11** (1958/59), Edinburgh Math. Notes No. 42 (misprinted 41) (1959), 1–2.

If $W = U + iV$ is the complex potential for a two-dimensional electrostatic system in the (x, y)-plane the electrostatic energy per unit length of the system is $\frac{1}{2}\varepsilon\iint dU\,dV$, where ε is the permittivity and the integration is over the image domain in the (U, V)-plane of the domain of the field in the (x, y)-plane. A corresponding result holds for the kinetic energy of a two-dimensional flow of perfect fluid.

W. D. Collins (Newcastle-upon-Tyne)

3419:

Sparnaay, M. J. Van der Waals forces and fluctuation phenomena. Physica **25** (1959), 444–454.

Author's summary: "An expression is given for the attraction between two electrically neutral systems each consisting of electric point charges which move at random inside spherical volumes. The calculation is based on the use of mean square values of density fluctuations in each system. The result and the method used are compared with Keesom's expression for the interaction between two dipoles and with the expression which is obtained for the interaction of two classical harmonic oscillators a large distance apart."

H. Mori (Kyoto)

3420:

Hoffman, W. C. (Editor). ★Statistical methods in radio wave propagation. Proceedings of a symposium held at the University of California, Los Angeles, June 18–20, 1958. Pergamcn Press, London-Oxford-New York-Paris, 1960. xiii + 334 pp. $14.00.

This book comprises the papers presented at a three-day meeting, held at the University of California, Los Angeles, in June, 1958. This symposium was the first occasion where statisticians and mathematicians working in the radio field were brought together with experimental radio workers, with the purpose of accomplishing a cross-fertilization and fusion of the mathematical techniques used. Random, unpredictable variations of the field strength of radio signals have long been a leading problem for radio propagation research. Only in the past ten years have refined statistical techniques, deriving principally from analogous results in random noise theory, come into general use. This book has ten essentially theoretical and eleven essentially experimental papers devoted to aspects of the problem ranging from sea clutter spectra to the extrapolation of spatial correlation functions. It should prove an indispensable reference for those in theoretical and experimental researches, gathering papers together which might otherwise have been scattered among a dozen learned journals.

S. A. Bowhill (University Park, Pa.)

3421:

Marcum, J. I. A statistical theory of target detection by pulsed radar. Trans. IRE **IT-6** (1960), 59–267.

This paper, which is of interest principally to radar design engineers, gives, in a series of 58 diagrams, the probability that a target will be detected by a pulse-type radar system. Most calculations of radar range are based on the concept of a sharp transition between certainty and uncertainty in the detection of a target, occurring at a given range. This paper takes complete account of the transition between certainty and uncertainty; it is, however, limited to a study of the effect of random noise at the receiver input rather than clutter or man-made static. *S. A. Bowhill* (University Park, Pa.)

3422:

Swerling, P. Probability of detection for fluctuating targets. Trans. IRE **IT-6** (1960), 269–308.

This report considers the probability of detection of a target by a pulsed search radar, when the target has a fluctuating cross section.

Formulas for detection probability are derived, and curves of detection probability vs. range are given, for four different target fluctuation models.

The investigation shows that, for these fluctuation models, the probability of detection for a fluctuating target is less than that for a non-fluctuating target if the range is sufficiently short, and is greater if the range is sufficiently long.

The amount by which the fluctuating and non-fluctuating cases differ depends on the rapidity of fluctuation and on the statistical distribution of the fluctuations. *S. A. Bowhill* (University Park, Pa.)

3423:

Wait, James R. A survey and bibliography of recent research in the propagation of VLF radio waves. Nat. Bur. Standards Tech. Note No. 58 (1960), 44 pp.

3424:

Haskind, M. D. Propagation of acoustic and electromagnetic waves in a half space. Akust. Ž. **5** (1959), 464–471 (Russian); translated as Soviet Physics. Acoust. **5** (1960), 476–484.

The problem as posed is to determine $\phi(x, y, z)$ satisfying the reduced wave equation $\Delta\phi+k^2\phi=0$ in the half-space $z\leq 0$, with boundary condition $\partial\phi/\partial n=V_n$ on a scattering and radiating surface S and $\partial\phi/\partial z-\nu\phi=0$ on $z=0$. The author introduces a function $f(x, y, z)$ which satisfies the same differential equation as ϕ and is connected with ϕ by the relation $\partial\phi/\partial z-\nu\phi=\partial f/\partial z$. The function f can be extended into the upper half-space so that one deals with a function that is regular and single-valued in all space outside the union of S and its reflection in the xy-plane. If f is known, then ϕ is given explicitly in terms of f and $\partial f/\partial z$ in the form of contour or surface integrals. *R. N. Goss* (San Diego, Calif.)

3425:

Bean, B. R.; Dutton, E. J. On the calculation of the departures of radio wave bending from normal. J. Res. Nat. Bur. Standards Sect. D **64D** (1960), 259–263.

The variation of the propagation direction (angular bending) of a radio wave in the troposphere follows from the profile of the refractive index as a function of the distance r to the centre of the earth. The integral representing this bending can be split in a contribution existing if this profile were given by an exponential dependence on the height $h=r-a$ above the earth's surface, and a correction term due to the deviation from such an idealized profile. The latter contribution turns out to be small if the parameters of the exponential function are chosen properly. The correction term can then be approximated by an expression depending on the true refractive indices at the ends of the path, and on the bending connected with the idealized profile. The approximation thus obtained is discussed and compared with other evaluations of the rigorous integral for the bending. *H. Bremmer* (Eindhoven)

3426:

Wait, James R. Radio wave propagation in an inhomogeneous atmosphere. Nat. Bur. Standards. Tech. Note No. 24 (1959), 20 pp.

This paper deals with the theory of atmospheric radio propagation, assuming a spherically curved stratified atmosphere. The Maxwell equations for such a medium are first reduced to a scalar wave equation, whereas the influence of the earth is accounted for by an impedance boundary condition at its surface. The paper reviews and extends previous work on the subject. Special attention is paid to Fock's approximative method according to which the wave equation is further reduced to a parabolic form, equation (8). The separation of variables, leading in the case of a homogeneous atmosphere [J. R. Wait and A. M. Conda, Trans. IRE. **AP-6** (1958), 348–359; MR **20** #7504] to an integral depending on Airy functions, is extended here to a general stratified atmosphere. The denominator of the integrand then depends on the solution of a "height-gain differential equation", viz. equation (42), for the radial part of the mode solutions. An application of the W.K.B. method, combined with a saddlepoint approximation, leads to the well-known derivation of the geometrical optics approximation. The analysis shows in particular the justification of the concept of an effective earth radius (enabling the reduction of the effect of the earth's curvature to that of introducing a proper modified refractive index), and the important role of the horizon distance of the transmitter. This geometrical parameter takes account of the main effects due to raising the transmitter and (or) receiver above the earth. An analysis for the determination of the horizon distance is worked out in detail in section 5 for an "exponential atmosphere" (the refractive index of which is given by an exponential function of the height). *H. Bremmer* (Eindhoven)

3427:

Ting, Lu. On the diffraction of an arbitrary pulse by a wedge or a cone. Quart. Appl. Math. **18** (1960/61), 89–92.

Author's summary: "By virtue of Green's theorem, it is shown that for the diffraction of an arbitrary two-dimensional incident pulse by a wedge of angle μ, the ratio of the resultant velocity potential to the corresponding value of the incident pulse at the corner of the wedge at any instant is $2\pi/(2\pi-\mu)$; and that for the diffraction of a three-dimensional pulse by a cone of solid angle ω, the ratio at the vertex of the cone is $4\pi/(4\pi-\omega)$." *E. T. Copson* (St. Andrews)

3428:

Ankerman, P. W.; Rubega, R. A. Bistatic scattering from totally reflecting flat plates. J. Acoust. Soc. Amer. **32** (1960), 478–481.

Authors' summary: "This paper deals with an investigation of the characteristics of the scattering of sound from a flat plate, five wavelengths long. The scatter patterns were calculated based on the simplifying assumption of total reflection and experimentally checked by echo ranging in air. Agreement was found to be reasonably good although some interesting deviations in the side-lobe structure were observed. Patterns calculated and experimentally obtained include bistatic backscatter patterns as well as the scatter patterns for a given incident wave."

3429:

Grinberg, G. A.; Pimenov, Yu. V. Diffraction of electromagnetic waves by a circular aperture in an ideally conducting plane. Ž. Tehn. Fiz. **29** (1959), 1206–1211 (Russian); translated as Soviet Physics. Tech. Phys. **4** (1960), 1106–1111.

A plane wave is assumed to impinge at normal incidence on an ideally conducting plane screen with a circular aperture. The integral equations for this diffraction problem are solved in an asymptotic manner assuming that ka is large (k the wave number and a the radius of the aperture). The results compare favorably with the exact solutions even when ka is relatively small (say 5) and represent a considerable improvement over the Huygens-Kirchhoff method which neglects the currents induced on the dark side of the screen.

E. W. Marchand (Rochester, N.Y.)

3430:

Müller, K. E. Untersuchung des Strahlungsverhaltens elliptischer Hohlleiter sowie der Möglichkeit zur Erzeugung eines zirkular polarisierten Strahlungsfeldes. Hochfrequenztech. Elektroak. **69** (1960), 140–151.

3431:

Kapica, P. L.; Fok, V. A.; Vaĭnšteĭn, L. A. Symmetrical electric oscillation of an ideally conducting hollow cylinder of finite length. Ž. Tehn. Fiz. **29** (1959), 1188–1205 (Russian); translated as Soviet Physics. Tech. Phys. **4** (1960), 1088–1105.

The electromagnetic problem considered corresponds to a flow of current along a cylinder (without ends) of length $2L$ and radius a, the current being uniformly distributed over the periphery. The problem is formulated in terms of an integral equation connecting the current density with the value of the single component of the vector potential, on the surface of the cylinder. The latter is a linear combination of a term corresponding to the external field and of the terms $\sin kz$ and $\cos kz$, where $k=2\pi/\lambda$ and λ is the wavelength. There are two free constants which are to be chosen so that the current vanishes at the extremes of the cylinder. The kernel of the integral equation is expanded in a Fourier cosine series of period $4L$ and the coefficients are obtained approximately under the assumptions $a/L \ll 1$, $ka^2/(2L) \ll 1$. The potential and the current are then expanded in various Fourier series and the integral equation is reduced to an infinite system of linear equations, the elements of the corresponding matrix each being an infinite sum. It is asserted that the matrix is nearly diagonal and the infinite system of equations, including the boundary conditions, is written in a form suitable for solution by iteration. The regularity of the system of equations is discussed. Comparison is made with the theory of thin radiators ($ka \ll 1$) according to E. Hallén [Nova Acta Soc. Sci. Upsal. (4) **11** (1938), no. 7, 1–44]. The computational method in this paper is not limited to the case of thin conductors, but no numerical results are given.

J. A. Morrison (Murray Hill, N.J.)

3432:

Huang, Hung-chia. Generalized theory of coupled local normal modes in multi-wave guides. Sci. Sinica **9** (1960), 142–154.

Author's summary: "In this paper a generalized theory of coupled local normal modes is developed, and a systematic mathematical method is introduced to solve problems of multi-coupled modes in waveguides with slowly varying parameters, which involve systems of linear differential equations with slowly varying coefficients. To illustrate the applicabilities of the method, the problem of bend with slowly varying curvature is solved by considering two and three coupled modes successively. For the two coupled modes case, our results agree with those of Louisell and Unger. Solution for the three coupled modes problem has not appeared in literatures heretofore. Further applications are discussed."

3433:

Papadopoulos, V. M. Wave propagation in a coaxial system. Quart. Appl. Math. **17** (1959/60), 423–436.

A semi-infinite flanged coaxial line with an infinite center conductor, all of circular cross-section, is excited from within the line by the radially symmetric dominant mode travelling toward the open end, where it is reflected into the line and propagated into the free half-space around the protruding infinite antenna. The resulting field is constructed inside the line by an infinite series of modes and in the free half-space by integral representations involving cylinder functions. When these are matched at the line's terminal, an infinite set of coefficients obeying an infinite set of linear equations determines the solution of the problem. This solution is worked out in limiting cases that illustrate properties both of a thin vertical antenna on a plane perfectly conducting earth and of a thick antenna fed by a low-impedance line. Some numerical results are included.

C. J. Bouwkamp (Eindhoven)

3434:

Medhurst, R. G. Harmonic distortion of frequency-modulated waves by linear networks. Proc. Inst. Elec. Engrs. Part. III **101** (1954), 171–181.

A method is derived for calculating the distortion of a frequency-modulated signal by a linear network. The network is assumed to have approximately linear phase and constant amplitude characteristics, the distortion being produced by small departures from these ideal characteristics. In this paper, expressions for phase and amplitude distortions have been derived for the case in which the modulating signal is a single-frequency sinusoid. The calculations are made by determining the distorting

effect produced upon each harmonic component of the modulated wave by the appropriate network characteristic, expressed as deviations from the ideal characteristics. A comparison of the results obtained by this method with those obtained by conventional operational methods is made. Tables of functions involved in making the calculations required by the derivation in this paper are included. *R. Kahal* (Washington, D.C.)

3435:

Medhurst, R. G. Fundamental and harmonic distortion of waves frequency-modulated with a single tone. Proc. Inst. Elec. Engrs. B **107** (1960), 155–164.

In an earlier paper [see preceding review] the author obtained an expression for the distortion of frequency-modulated waves by a linear network having small departures from linear phase and constant amplitude frequency characteristics. The modulating signal was assumed to be a single-frequency sinusoid. In this paper a new approximate formula has been developed by an extension of the earlier method. In particular, in certain circumstances the assumption of small distortions, even when the deviation of the phase characteristic from linear behavior is not small, remains valid. Thus the use of the extended theory together with direct Fourier analysis or quasi-stationary theory permits evaluation of distortion values over extended range of parameter values. Several numerical examples are treated, with particular attention to distortion of the amplitude and phase of the fundamental. *R. Kahal* (Washington, D.C.)

3436:

Horvat, Radoslav D. Integral theorems of electric network characteristics. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 28 (1959), 15 pp. (Serbo-Croatian. English summary)

One considers the integral relations between the real and imaginary part of the network characteristics which are analytic in the right half plane of complex frequency and without poles at the origin and infinity. By means of Cauchy's theorem and convenient functions several theorems are deduced. It is shown that the complex functions must satisfy these relations if they represent network characteristics. First, the reactance and resistance integral theorems are derived. Further, the author shows that the potential series coefficients of the real and imaginary part of the network characteristic for frequencies lower than the minimum value of the modulus of its poles are represented by the integrals determined in the first case by the imaginary and in the second by the real part of the characteristic on the real frequency axis.

D. P. Rašković (Belgrade)

3437:

Cunningham, W. J. Oscillating systems with one variable element. J. Franklin Inst. **269** (1960), 81–92.

The oscillations of a passive electric circuit composed of a capacitance, inductance, and resistance in series are analyzed mathematically. It is assumed that the capacitance is caused to change monotonically with time and that the resistance element is small enough to allow oscillation in the circuit. The circuit is analyzed by the use of the WKBJ method, and by the use of energy considerations. It is shown that the changing capacitance produces changes in both the instantaneous frequency and the amplitude of oscillation. Analogous mechanical systems and the practical implications of the analysis are considered. *L. A. Pipes* (Los Angeles, Calif.)

3438:

Weinzweig, A. I. The Kron method of tearing and the dual method of identification. Quart. Appl. Math. **18** (1960/61), 183–190.

J. P. Roth [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 518–521, 599–600; MR **17**, 536] was presumably the first to give a mathematically detailed treatment of Kron's method of tearing. This paper gives additional insight into the method by considering a somewhat more general situation than the usual one. As a problem in linear algebra, it can be formulated thus: one is given vector spaces X, Y; direct sum decompositions $X=X_1\oplus X_2$, $Y=Y_1\oplus Y_2$; linear transformations $\partial: X\to Y$, $\lambda: X\to X^*$ (the dual space) such that $\partial[X_i]\subset Y_i$, and vectors in X_2, X_1^*, Y_1, Y_2^*. One is to find vectors x, x^*, y, y^* whose projections into various subspaces are the given vectors and such that $\partial x=y$ and $\lambda x=x^*+\partial^* y^*$. Granting a condition on λ the solution exists and is essentially unique. Letting $\lambda(C_1): C_1\to C_1^*$ be the linear transformation defined from λ and a direct sum decomposition $X=C_1\oplus C_2$, the solution of the problem involves inverting $\lambda(X_1')$ where X_1' is the null space of ∂ restricted to X_1, $X=X_1'\oplus X_2'$. If the inverse of $\lambda(\bar{X}_1)$, where $X=\bar{X}_1\oplus\bar{X}_2$ and $\bar{X}_1\supset X_1'$, is known it may be used to invert $\lambda(X_1')$ (an additional "connection transformation" must be inverted). Granting a suitable choice of bases, the problem is to invert appropriate matrices, and in this context tearing uses the topology of the network to find a decomposition of X for which the appropriate matrices are strongly diagonal. The author notes that tearing is more appropriate for the admittance form of the network problem and that there is a dual procedure appropriate to the impedance form. The dual procedure is called the method of identification and, in a sense, is the one actually described above.

J. B. Giever (University Park, N.M.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 3311, 3316, 3317, 3505.

3439:

Siegel, Robert. Heat transfer for laminar flow in ducts with arbitrary time variations in wall temperature. J. Appl. Mech. **27** (1960), 241–249.

L'auteur considère le problème du transfert thermique, en convection forcée, pour un écoulement laminaire dans un canal dont la température de paroi varie dans le temps, à partir d'un régime établi, suivant une loi donnée.

Le problème fondamental traité correspond aux conditions suivantes: à l'instant $\theta=0$, le fluide et la paroi sont à la température uniforme $t=0$; pour $\theta>0$, $x\geqq 0$ (x abscisse le long du canal), la température de paroi est fixée à la valeur $t=1$. Ce problème est étudié dans les deux cas d'un canal circulaire et d'un canal rectangulaire.

Ce problème de diffusion en régime non permanent est remplacé par un problème plus simple, dans lequel la

solution, durant le régime transitoire, est astreinte à vérifier seulement une forme intégrée, dans la section, de l'équation de la diffusion. Ce n'est que pour le régime asymptotique que la solution ainsi formée vérifie les équations exactes.

Sous cette forme approchée, le problème est résolu de deux manières différentes: dans le cas du canal circulaire, il est réduit à une équation aux dérivées partielles du premier ordre, traitée par la méthode des caractéristiques, et intégrée numériquement, sur une I.B.M. 653; dans le cas du canal entre plaques, la solution est développée, d'une manière approchée, en série de polynômes trigonométriques, les calculs de matrices étant toujours effectués sur I.B.M. 653.

Les résultats numériques sont donnés et discutés.

Il est montré comment le problème ainsi traité pourrait être généralisé à une variation arbitrairement donnée de la température de paroi, à partir d'un régime établi avec transfert. *R. Gerber* (Grenoble)

3440:

Baxter, C. B.; Davies, D. R. Heat transfer by laminar flow from a rotating spherical cap at large Prandtl numbers. Quart. J. Mech. Appl. Math. **13** (1960), 247–250.

The approximate method of Davies [same J. **12** (1959), 211–221; MR **21** #4003] for calculating heat transfer from a rotating disk is applied to the case of a spherical cap. The inner part of Howarth's Pohlhansen solution [Philos. Mag. (7) **42** (1951), 1308–1315; MR **13**, 506] for the velocity distribution is used to reduce the energy equation at high Prandtl numbers (i.e., thin thermal boundary layers) to the equation solved in the previous paper, and estimates are made of the ratio of heat transfer from a spherical cap to that from a rotating disk of equal area; for a cap of 80° semi-angle (the limit of Howarth's solution), the ratio has dropped from unity only to about 0.79. *D. A. Spence* (Pasadena, Calif.)

3441:

Andriankin, È. I. Propagation of thermal waves from the boundary of two media. Prikl. Mat. Meh. **23** (1959), 991–992 (Russian); translated as J. Appl. Math. Mech. **23**, 1420–1423.

Author's summary: "The problem of the propagation of heat from a plane boundary, and the decay of the discontinuity of temperature is considered, taking account of the change in phase of the material. An evaluation is carried out of the energy propagated into a medium with small thermal conductivity upon instantaneous evolution of heat at a point on the boundary of separation of two media."

3442:

Boley, Bruno A. Upper bounds and Saint-Venant's principle in transient heat conduction. Quart. Appl. Math. **18** (1960/61), 205–207.

A domain D is contained in a domain D^* and the boundary surfaces of the two domains have a portion S in common. For the heat equation $T_t = k\nabla^2 T$, let G and G^* denote Green's functions for the domains D and D^*. An inequality for the normal derivatives of G and G^* on S is pointed out. When D^* is a half space $x>0$ and the temperatures over the boundaries are zero except on a disk S_0 on the plane $x=0$, and when initial temperatures are zero, the inequality provides a useful upper bound for temperatures $|T|$ in D. The upper bound is employed to show in what respects the temperatures at points of D distant from S_0 are only slightly influenced by the thermal disturbance on S_0. *R. V. Churchill* (Ann Arbor, Michigan)

3443:

Poritsky, H.; Powell, R. A. Certain solutions of the heat conduction equation. Quart. Appl. Math. **18** (1960/61), 97–106.

The solution $T = T_n(x, t)$ of the heat equation $T_t = kT_{xx}$ in the domain $x>0$, $t>0$ under the conditions $T(x, 0)=0$, $-KT_x(0, t) = h(t)$, is examined here when $h(t) = t^n/\Gamma(n+1)$. When $n = 0, 1, 2, \cdots$, recurrence relations between T_n and their derivatives, and between certain polynomials involved in a formula for T_n, are obtained. A similar study is made of the temperature functions $T_{n-1/2}(x, t)$. *R. V. Churchill* (Ann Arbor, Mich.)

3444:

Hassan, H. A. On a solution to the unsteady laminar boundary layer. J. Aero/Space Sci. **27** (1960), 474–476.

3445:

Campbell, W. F. Periodic temperature distribution in a two-layer composite slab. J. Aerospace Sci. **27** (1960), 633–634.

3446:

Hirschfelder, Joseph O.; McCone, Alan, Jr. Theory of flames produced by unimolecular reactions. I. Accurate numerical solutions. Phys. Fluids **2** (1959), 551–564.

Es werden die Differentialgleichungen für die laminare, stationäre eindimensionale Flamme einer einmolekularen Reaktion unter Berücksichtigung der thermischen Diffusion, der Wärmestrahlung, der Wirkung äusserer Kräfte und der Druckänderung aufgestellt. Die Gleichungen werden numerisch integriert, wobei die wesentlichen Parameter systematisch verändert werden. Diese exakten Lösungen sind zur Kontrolle physikalischer Näherungsmodelle gedacht. Mit diesen exakten Lösungen wird unter anderem der Einfluss des Flammenhalters auf die Flammenfortpflanzungsgeschwindigkeit hergeleitet. Es wird gezeigt, dass die bei grösstem Wärmeentzug durch den Flammenhalter auftretende Flammengeschwindigkeit die kleinste stabile Flammengeschwindigkeit ist. *L. Speidel* (Mülheim)

3447:

Hirschfelder, Joseph O. Theory of flames produced by unimolecular reactions. II. Ignition temperature and other types of approximations. Phys. Fluids **2** (1959), 565–574.

Im Anschluss an die exakte Berechnung einer laminaren, stationären Flamme einer einmolekularen Reaktion [siehe vorangehende Besprechung] wird die Eintzündungstemperatur derartiger Flammen berechnet. Es wird weiter ein Näherungsverfahren zur Bestimmung der Flammenfortpflanzungageschwindigkeit entwickelt, das die Berechnung dieser Grösse bis auf wenige Prozent genau gestattet. *L. Speidel* (Mülheim)

3448:

Senior, D. A. A theoretical treatment of combustion in a spherical underwater gas bubble. Proc. Roy. Soc. London. Ser. A **251** (1959), 493–503.

Author's summary: "The combustion equations and the equation of motion of the surface of the bubble are formulated on the assumptions that the gases are ignited at their centre and that the flame front travels outward in the form of a sphere. The equations are integrated to yield gas pressure and bubble radius as functions of time. Provided that radial displacements are small, the flame speed can be assumed constant. The formulation is then simple and the equations can be integrated in closed form to give general non-dimensional solutions. These show that pressure and radius increase at an increasing rate until combustion is complete. Radial oscillations ensue whose amplitude in proportion to the initial radius decreases with increase in the ratio of combustion time to oscillation period. For large radial displacements, the formulation is less simple as finite expansion of the burnt gases at the expense of the unburnt gases is involved, which enhances the flame speed. The effect has already been considered for combustion in a closed spherical vessel [Flamm and Mache, Akad. Wiss. Wien. Math.-Nat. Kl. S.-B. Abt. IIa **126** (1917), 9–44], and an important approximate relation between the fraction of gas burned and the pressure rise has been derived: $n=(p-P_0)\cdot(P-P_0)^{-1}$, where n is the fraction of gas burned, p is the corresponding pressure and P and P_0 are respectively the final and initial pressures. The analysis of Flamm and Mache, which is in any case not entirely satisfactory, cannot be used if the total gas volume varies, as in the present instance. An energy method has, however, been found to give the required relation (and incidentally to afford a simpler, more rigorous proof of the closed vessel relation and a measure of its accuracy). The equations require numerical integration and some typical results are given in graphical and tabular form. These show two features which are absent when displacements are small: as in closed vessels the combustion time is shorter than that for constant flame speed (typically, by a factor of 3); and, during combustion, pressure fluctuations develop which arise from the interaction of combustion and expansion each of which leads to an acceleration of the other."

QUANTUM MECHANICS

See also 3153, 3175, 3176, 3275, 3276, 3277, 3537, 3541, 3555, 3556.

3449:

Schwinger, Julian. The algebra of microscopic measurement. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1542–1553.

The author gives an outline of a foundation of the quantum mechanical theory of measurement, starting from physical statements about the nature of the process of measurement. By confining his attention to the simple case in which observables can take only a finite number of different values, the author avoids all technical mathematical difficulties. The analysis goes beyond the well-known analysis of von Neumann [*Mathematical foundations of quantum mechanics*, Princeton Univ. Press, Princeton N.J., 1955; MR **16**, 654; Chapters II and IV] in treating measurements, $M(a', b')$, in which given systems are rejected if they do not have the value, b', of an observable b and if accepted are transformed into systems which have the value a' of the observable a. (In the quantum mechanical formalism $M(a', b')$ is the matrix $\psi_{a'}\otimes\bar{\psi}_{b'}$, where $\psi_{a'}$ is the state vector satisfying $a\psi_{a'}=a'\psi_{a'}$ and similarly for $\psi_{b'}$.) The author assumes $M(a', b')M(c', d')=\langle b'|c'\rangle M(a', d')$ where $\langle b'|c'\rangle$ is a complex number. {While this equation is true in quantum mechanics, in the reviewer's opinion it does not have a direct physical meaning within the author's framework because he does not give a meaning to a complex number times a measurement.} Given this equation and $\langle b'|c'\rangle=\overline{\langle c'|b'\rangle}$, the author gives an elegant and simple deduction of the ordinary formalism of quantum mechanics.

A. S. Wightman (Princeton, N.J.)

3450:

Vanagas, V. V.; Čiplis, I. V. [Ciplys, J.]. Transformation matrix and its connection with j-symbols. Trudy Akad. Nauk Litov. SSR. Ser. B **3 (15)** (1958), 17–33. (Russian. Lithuanian summary)

The problem of reduction of Kronecker products of representations of the 3-dimensional rotation group has given rise to an extensive specialized literature, based on the vector coupling model. The authors discuss the connection between the matrices which perform this reduction (Clebsch-Gordan and Racah matrices) and the quantities known as Wigner's j-symbols. {For similar considerations see A. R. Edmonds, *Angular momentum in quantum mechanics*, Princeton Univ. Press, Princeton, N.J., 1957; MR **20** #2201; chapters 3 and 7.} Diagrams and tables are given for the special case of coupling of 4 and of 5 vectors.

E. L. Hill (Minneapolis, Minn.)

3451:

Levinson, I. B. [Levinsonas, J.]; Čiplis, I. V. [Ciplys, J.]. Recursive construction of diagrams of j-symbols. 15 j-symbols. Trudy Akad. Nauk Litov. SSR. Ser. B **1 (13)** (1958), 3–9. (Russian. Lithuanian summary)

Methods are discussed for the construction of diagrams for Wigner j-symbols with $3n$ parameters from those with $3n-3$ parameters. The method is applied to the 15 j-symbols which are not products of simpler symbols. Symmetry properties and formulas for calculation of the symbols are given.

E. L. Hill (Minneapolis, Minn.)

3452:

Fairlie, D. B. The Fredholm solution as the limit for a sum of separable potentials. Proc. Cambridge Philos. Soc. **56** (1960), 182–185.

The author obtains an exact solution to the problem of scattering in non-relativistic quantum mechanics by a sum of n separable potentials. As $n\to\infty$ the Fredholm solution for the integral Schrödinger equation is obtained.

J. C. Polkinghorne (Cambridge, England)

3453:

Konisi, Gaku; Ogimoto, Takesi. Quantum theory in pseudo-Hilbert space. Progr. Theoret. Phys. **20** (1958), 868–875.

A Lorentz invariant theory of quantum electrodynamics is obtained in a Hilbert space with an indefinite metric introduced by S. Gupta [Proc. Phys. Soc. **63** (1950), 681–691; Canad. J. Phys. **35** (1957), 961–968; MR **12** 67; **19**, 711]. The noninvariance of the older theory is avoided by distinguishing contravariant and covariant transformation properties of second rank operators in the space considered. The operator η transforming the electromagnetic four-potential A_μ into its Hermitian conjugate A_μ^* ($A_\mu = \eta^{-1} A_\mu^* \eta$) is adopted as the metric used in raising and lowering the suffixes of the operators and of the state vectors. In showing its Hermitian property it must be noted that neutrality of the field implies a dependance between A_μ and A_μ^* [see footnote to the paper reviewed below].

There seems to be an incorrect criticism of spinor formalism based on the assumption of equivalence between the views that either the γ matrices or the wave function must be transformed to ensure invariance of the wave equation. The former view is repudiated by Bade and Jehle [Rev. Modern Physics **25** (1953), 714–728; MR **15**, 162] since it violates the spirit of the theory of relativity. *A. H. Klotz* (Newcastle-upon-Tyne)

3454:

Konisi, Gaku; Ogimoto, Takesi. Quantum theory in pseudo-Hilbert space. II. Progr. Theoret. Phys. **21** (1959), 727–730.

Using the formalism developed in the previous paper it is shown that Lorentz invariant theory obtains also in the case of interaction fields.
A. H. Klotz (Newcastle-upon-Tyne)

3455:

Gel'fand, I. M.; Yaglom, A. M. Integration in functional spaces and its applications in quantum physics. J. Mathematical Phys. **1** (1960), 48–69.

Translation of the paper in Uspehi Mat. Nauk **11** (1956), no. 1 (67), 77–114 [MR **17**, 1261].

3456:

Kyrala, Ali. Matter and point set theory. Phys. Rev. (2) **117** (1960), 1409.

Author's summary: "A brief theoretical exposition of the concept of matter as a nowhere dense perfect set of positive measure is developed starting from the observation that matter is largely vacuous in all scales from subnuclear to extragalactic."

3457:

Aaron, Ronald; Klein, Abraham. Convergence of the Born expansion. J. Mathematical Phys. **1** (1960), 131–138.

This paper shows that the Born series for scattering by a spherically symmetric potential in n dimensions will converge for sufficiently high values of the energy provided that the potential satisfies a set of rather wide conditions. These are the generalizations of the conditions found previously by Zemach and Klein [Nuovo Cimento (10) **10** (1958), 1078–1087; MR **21** #580] for three dimensions. *J. C. Polkinghorne* (Cambridge, England)

3458:

Volkov, D. V.; Peletminskiĭ, S. V. On the Lagrangian formalism for spin variables. Ž. Èksper. Teoret. Fiz. **37** (1959), 170–178 (Russian); translated as Soviet Physics. JETP **10** (1960), 121–126.

Authors' summary: "It is shown that a change in the class of allowable variations in the Schwinger variation principle makes it possible to include the spin variables in the general Lagrangian formalism in both the nonrelativistic and relativistic cases."
A. Klein (Philadelphia, Pa.)

3459:

Maslov, V. P. On the transition from quantum mechanics to classical in the multidimensional case. Uspehi Mat. Nauk **15** (1960), no. 1 (91), 213–219. (Russian)

3460:

Weidlich, Wolfgang. Zur Interpretation der Quantenmechanik. Z. Naturforschg. **15a** (1960), 651–654.

Author's summary: "Es werden allgemeine Bedingungen angegeben für den Zusammenhang eines möglichen Parameterraumes, in dem eine determinierte Beschreibung physikalischer Systeme stattfinden soll, mit dem Hilbert-Raum der Quantenmechanik. Vorausgesetzt ist die Gültigkeit der quantentheoretischen Messaxiome und Bewegungsgleichungen."

3461:

Spruch, Larry; Rosenberg, Leonard. Upper bounds on scattering lengths for static potentials. Phys. Rev. (2) **116** (1959), 1034–1040.

This is the first paper to point out explicitly that the scattering length is characterized as the minimum of a certain quadratic functional under rather general conditions. In this paper scattering by a static, spherically symmetric potential is treated. Consider, for example, the S-wave equation: $\mathscr{L}u \equiv d^2u/dr^2 + W(r)u = 0$, $0 < r < \infty$, corresponding to zero angular momentum and zero energy. If $W(r) \to 0$ rapidly for $r \to \infty$, the solution u of $\mathscr{L}u = 0$ with $u(0) = 0$ has the asymptotic form $u \sim \text{const.}\,(A - r)$ for $r \to \infty$. A is the scattering length and related to the zero energy cross section σ by $\sigma = 4\pi A^2$. It is known (the Kohn variation principle) that A is the stationary value of the quadratic functional $K[u_t] \equiv A_t - \int_0^\infty u_t \mathscr{L} u_t \, dr$, where the trial function u_t is assumed to satisfy the boundary conditions $u_t(0) = 0$ and $u_t \sim A_t - r$, $r \to \infty$, with a constant A_t. The authors now prove that this stationary value is actually the absolute minimum so that $K[u_t]$ gives an upper bound on A, provided there exists no bound state (that is, if $\mathscr{L}u + \lambda u = 0$ with $u(0) = 0$ has no eigenvalue $\lambda < 0$ with an L_2 eigenfunction u, or equivalently, if the solution u of $\mathscr{L}u = 0$ has no zero other than $r = 0$). This minimum principle is very useful in approximate calculations of A, as is illustrated by numerical examples in neutron-proton scattering. It is remarked that the normalization $u_t \sim A_t - r$ is essential for the validity of the minimum principle, other normalizations such as $u_t \sim 1 - r/A_t$ being inadequate. The above results are extended to the scattering lengths for higher angular momenta. *T. Kato* (Tokyo)

3462:

Spruch, Larry; Rosenberg, Leonard. Upper bounds on scattering lengths for compound systems: n-D quartet scattering. Phys. Rev. (2) **117** (1960), 1095–1102.

The fact that the Kohn variational method provides an upper bound on the scattering length, first proved by the same authors in the preceding paper in the case of scattering by a static potential, is shown to be valid even for scattering by a compound system under rather general conditions; an essential condition is that no bound state, compatible with the prescribed angular momentum, parity, etc., should exist for the total system. A general formulation of the principle is presented for a many-particle system, and a simpler and more direct proof of the minimum property is given than that in the previous paper. The theory is applied to the neutron-deuteron quartet scattering length, and various numerical results so far obtained by different authors are analysed in detail from this point of view. *T. Kato* (Tokyo)

3463:

Rosenberg, Leonard; Spruch, Larry; O'Malley, Thomas F. Upper bounds on scattering lengths when composite bound states exist. Phys. Rev. (2) **118** (1960), 184–192.

The minimum principle for the scattering length, proved in the two preceding papers, is extended to the case in which there are a finite number of bound states of the total system. The results are stated in terms of the scattering by a static potential. In the presence of N bound states, the Kohn variational expression $K[u_t]\equiv A_t-\int_0^\infty u_t\mathscr{L}u_t dr$ for the scattering length A need not give an upper bound on A (for the notation see #3061) since the second variation $\delta^2K[u_t]=-2\int \delta u_t\mathscr{L}\delta u_t dr$ is not positive-definite. But the authors construct a modified functional $K'[u_t]\equiv K[u_t]+\sum_{i=1}^N (1/\varepsilon_i)(\int v_i\mathscr{L}u_t dr)^2$ which has A as its absolute minimum at $u_t=u=$(the exact solution), thus providing an upper bound of A. Here v_i, $i=1, \cdots, N$, are any N normalizable functions such that $\int v_i\mathscr{L}v_k dr=-\varepsilon_i\delta_{ik}$ with $\varepsilon_i>0$, and the trial function u_t should be normalized appropriately ($u_t\sim A_t-r$ for $r\to\infty$). Mathematically, the problem consists in converting a quadratic functional with a non-positive-definite principal part into one with a positive-definite principal part and with the same stationary value. The extension of the result to more general cases (scattering by a compound system, consideration of the Pauli principle, etc.) is straightforward. As an application, some earlier numerical calculations on electron-hydrogen scattering are analysed in the light of the present result, and upper bounds on A are obtained by a trivial conversion of the data. *T. Kato* (Tokyo)

3464:

Martin, A. On the analytic properties of partial wave scattering amplitudes obtained from the Schrödinger equation. Nuovo Cimento (10) **14** (1959), 403–425. (French and Italian summaries)

The analytic properties of the S 'matrix' $S(k)$ are studied for the Schrödinger equation. For finite range, but not necessarily local, potentials and spherical symmetry ($l=0$), S is analytic in $\operatorname{Im}(k)>0$, has poles for $\operatorname{Im}(k)<0$ corresponding to zeros in $\operatorname{Im}(k)>0$. Poles near the real axis in $\operatorname{Im}(k)<0$ occur corresponding to decaying bound states.

Related results are derived for $l\neq 0$.

For $l=0$ and potentials of infinite range but decreasing faster than an exponential, poles on a part of the imaginary axis corresponding to bound states are found.

Examples discussed are sum of exponential, oscillating, and Yukawa potentials. *C. Strachan* (Aberdeen)

3465:

Moffat, John W. Resonance behaviour of scattering amplitudes in dispersion relations. Nuclear Phys. **18** (1960), 75–80.

The author writes a dispersion relation for the scattering amplitude $T(W)$ and its inverse $T^{-1}(W)$, assuming the latter has no complex poles. By using the method proposed by Schwinger [Ann. Physics **9** (1960), 169–193; MR **22** #2362] in his treatment of unstable particles, a phase angle $\theta(W)$ is defined, by which $T(W)$ is determined. Discontinuities in θ give rise to resonances in $T(W)$; these are shown to be of the Breit-Wigner form. Using the optical theorem another well-known result is obtained: the shape of the resonance in the total cross-section can be determined if the location is known.

R. F. Streater (Princeton, N.J.)

3466:

Pearlstein, L. D.; Klein, A. Theory of the photodisintegration of the deuteron. Phys. Rev. (2) **118** (1960), 193–211.

A method developed by Klein and Zemach [Phys. Rev. (2) **108** (1957), 126–138; MR **20** #7558] is used to derive a formally exact expression for the photodisintegration of the deuteron. The nucleon-nucleon interaction is treated using an expansion in numbers of mesons exchanged, and the resulting series is broken off after the terms corresponding to one meson being exchanged. The theory is applied in the energy range 100–400 Mev. Good agreement with the experimental total cross-section is obtained if both hard-core and tensor-force effects are included. *D. ter Haar* (Oxford)

3467:

Watanabe, Shigueo. Low energy elastic scattering of nucleons by deuterons. Nuclear Phys. **14** (1959/60), 429–437.

Author's summary: "The elastic scattering of nucleons by deuterons at low energies is discussed. The interaction forces between two nucleons include the central exchange and spin-orbit ones. The inclusion of the two-body spin-orbit force leads to a set of coupled integro-differential equations." *P. Chevallier* (Strasbourg)

3468:

Wataghin, G. Causality complementarity and S-matrix formalism in a non-local relativistic theory of fields. Nuovo Cimento (10) **14** (1959), 1157–1165. (Italian summary)

Author's summary: "A new formalism of a non-local theory is suggested permitting one to substitute the description of point-interactions and point particles by interactions taking place in certain 4-dimensional domains and by an approximate description of propagation of fields representing extended physical particles. The conditions

of relativistic invariance and of macroscopic causality are compatible with the above limitations because the interactions can be described by the S-matrix in p-space and studied independently from the propagation problem."

3469:

Zlatev, I. S.; Isaev, P. S. Dispersion relations for the virtual Compton effect. Ž. Èksper. Teoret. Fiz. **37** (1959), 728–734 (Russian); translated as Soviet Physics. JETP **10** (1960), 519–523.

Authors' summary: "Dispersion relations for physical amplitudes have been derived by the Bogolyubov method in the center-of-mass system for electron bremsstrahlung and for pair production by a photon in the field of a nucleon, accurate to lowest order in e."

3470:

Källén, G.; Wightman, A. The analytic properties of the vacuum expectation value of a product of three scalar local fields. Mat.-Fys. Skr. Danske Vid. Selsk. **1** (1958), no. 6, 1–58.

Authors' summary: "From the general requirements of Lorentz invariance, resonable mass spectrum shape, and local commutativity it follows that the vacuum expectation value of a product of field operators is the boundary value of an analytic function. A corresponding statement holds for the Fourier transform of the retarded commutator and for the time ordered product of the fields. For the special case of three scalar fields, it is shown that, in general, the domains of analyticity obtained in x-space and in p-space are identical. This domain is explicitly computed and shown to be bounded by pieces of analytic hypersurfaces. These surfaces intersect in corners which are of such a kind that the domain is not a natural domain of analyticity. The holomorphy envelope of this domain is computed using only elementary methods. The result turns out also to be bounded by pieces of analytic hypersurfaces."

M. Cini (Rome)

3471:

Gribov, V. N. The causality condition and spectral representations of Green's functions. Soviet Physics. JETP **34 (7)** (1958), 903–909 (1310–1318 Ž. Èksper. Teoret. Fiz.).

The author wants to find a spectral representation for the time ordered products of three field operators imposing only the conditions of a reasonable mass spectrum and local commutativity. He first notes that the non-time-ordered product can be written as

$$\langle 0|\varphi(x_1)\varphi(x_2)\varphi(x_3)|0\rangle = (2\pi)^{-9}\iint dp_1dp_3 \exp(ip_1(x_1-x_2)+ip_3(x_2-x_3)) \times \theta(p_1)\theta(p_3)\rho(p_1^2;q^2;p_3^2).$$

$$q^2 = (p_1-p_3)^2,$$

where the function ρ is different from zero only when both the vectors p_1 and p_3 are time like. For equal times $x_{1_0}=x_{2_0}$ one has

$$\langle 0|\varphi(x_1)\varphi(x_2)\varphi(x_3)|0\rangle = \langle 0|\varphi(x_2)\varphi(x_1)\varphi(x_3)|0\rangle,$$

which implies

$$\int dp_{1_0}\theta(p_1)\rho(p_1^2;q^2;p_3^2) = \int dp_{1_0}\theta(p_1)\rho(\bar{p}_3-\bar{p}_1)^2-(p_{1_0}^2;\bar{p}_1^2-(p_{3_0}-p_{1_0})^2;p_3^2).$$

A similar condition is obtained for $x_{2_0}=x_{3_0}$. The argument then goes: "These conditions are satisfied if we write $\rho(p_1^2;q^2;p_3^2)$ in the form

$$\theta(p_1)\theta(p_3)\rho(p_1^2;q^2;p_3^2) = \int \theta(k_1)\theta(k_2)\theta(k_3)f(-k_1^2;-k_2^2;-k_3^2)dl$$

$$k_1 = \tfrac{1}{2}(-p_1+l+p_3);\; k_2 = \tfrac{1}{2}(p_1-l+p_3);\; k_3 = \tfrac{1}{2}(p_1+l-p_3),$$

and postulate that $f(-k_1^2;-k_2^2;-k_3^2)$ is a symmetric function of its argument which vanishes if any of them is less than zero." The rest of the paper is based on this representation of ρ. Taken literally, the above statement is correct, but the reviewer wants to mention that this is not the only way in which the integral equations above can be fulfilled. In fact, it has been shown that the representation above leads to a domain of analyticity of the time ordered product in p-space which is larger than the domain implied by local commutativity (causality) alone. [Cf. in this connection Appendix III of the paper by the reviewer and A. Wightman, reviewed above]. Therefore, the integral representations of the "three point function" obtained in the paper reviewed here are not rigorous consequences of local commutativity alone.

G. Källén (Lund)

3472:

Gribov, V. N. Renormalization of the vertex part in pseudoscalar meson theory. Soviet Physics. JETP **36 (9)** (1959), 384–387 (554–559 Ž. Èksper. Teoret. Fiz.).

The author investigates the properties of the vertex function in pseudoscalar meson theory under the assumption that this function can be represented with the aid of an integral representation previously suggested by the author [same Ž. **34** (1958), 1310–1318; see preceding review] and by others. This representation implies that the analytic function related to the x-space properties of the vacuum expectation value of the product of the meson field and two nucleon fields is regular analytic in the product of the three cut planes. As is well known, this big domain of regularity does not follow from general properties of the theory as do Lorentz invariance, local commutativity and reasonable mass spectrum of the theory. The author remarks that his representation is valid in perturbation theory and that it is not excluded that it can be proved with the aid of other general properties not mentioned above. {Reviewer's remark: It has been shown [G. Källén and A. Wightman, #3470; appendix III, especially Eq. (A59)] that it is possible to obtain a function which does not have this big analyticity domain in x-space by taking the first nontrivial term in the so-called Ward-theory treated by perturbation theory and integrating it over a suitable weight function. Therefore, the author's assertion that his representation always follows in perturbation theory should be accepted with

some reservation.} Starting from this basic assumption the author investigates the connection between his representation and various renormalization constants. In particular, it is shown that the high energy behaviour of the vertex function is intimately related to the renormalization constants. Finally, the author discusses the relation between his representation and the two-point function.

G. Källén (Lund)

3473:

Bonnevay, G. La diffusion méson-nucléon dans l'état *S* et l'interaction méson-méson en théorie de la source fixe. I. Diffusion par la source. Nuovo Cimento (10) **14** (1959), 593–611. (Italian summary)

Author's summary: "On discute les valeurs des déphasages S de la diffusion méson-nucléon déduites de la théorie de la source fixe considérée comme limite de la théorie relativiste pseudoscalaire indépendante de charge, la constante de couplage étant déterminée par la diffusion P. Les amplitudes de diffusion ainsi obtenues sont trop grandes et toutes deux de même signe. Elles serviront de point de départ à un prochain travail dans lequel il sera tenu compte de la diffusion du méson incident par le nuage du nucléon cible."

P. Chevallier (Strasbourg)

3474:

Steinmann, O. Wightman-Funktionen und retardierte Kommutatoren. II. Helv. Phys. Acta **33** (1960), 347–362. (English summary)

[For part I, see the author, same Acta **33** (1960), 257–298; MR **22** #2372.]

Author's summary: "The consequences of the basic postulates of quantum field theory (Lorentz-invariance, locality, stability of the vacuum) for the retarded products are investigated by considering their connection with Wightman's functions. Necessary and sufficient conditions for the existence of a Wightman function corresponding to a prescribed r-function are given. The Fourier transform $\tilde{r}(p_1, \cdots, p_n)$ of r is a boundary value of a function $\tilde{r}(k_1, \cdots, k_n)$ regular in a domain $\overline{\mathfrak{R}}_n$. $\overline{\mathfrak{R}}_n$ is constructed by a recursive procedure. Other boundary values $\tilde{g}_\mu(p_1, \cdots, p_n)$ of this function are considered. They have to fulfill a set of linear identities of four and twelve terms respectively."

3475:

Yoshimura, Tetz. On high energy limit of fermion-fermion interaction. Progr. Theoret. Phys. **23** (1960), 569–575.

The author investigates the connection between the high energy behaviour of the vertex part and the one body propagator in a field theory with interacting Fermion fields. In particular, he argues that it is impossible to draw any reliable conclusion about these high energy behaviours with the aid of truncated integral equations for the propagators. This problem is of interest in connection with the attempts by Landau and collaborators to investigate the high energy behaviour of field theories [cf. e.g., L. D. Landau, A. A. Abrikosov and I. M. Halatnikov, Dokl. Akad. Nauk SSSR **95** (1954), 497–500, 773–776, 1177–1180; MR **16**, 315, 316; and later papers].

G. Källén (Lund)

3476:

Yoshimura, Tetz. Asymptotic theory of interacting fields without Hamiltonian. Progr. Theoret. Phys. **23** (1960), 576–582.

This paper continues the author's previous work [cf. the preceding review] and extends the investigation to cases of interactions which cannot be handled with the aid of a Hamiltonian formalism. The conclusion of the paper is that there exists a faint mathematical possibility for these theories to be finite but that one has to hope for rather special circumstances for this to be the case.

G. Källén (Lund)

3477:

Segal, I. E. Foundations of the theory of dynamical systems of infinitely many degrees of freedom. I. Mat.-Fys. Medd. Danske Vid. Selsk. **31**, no. 12, 39 pp. (1959).

The well-known "divergences" of quantum field theory have led physicists to examine the foundations of the subject with a more critical eye, and various of them have suggested that some modification of the conventional Hilbert space formalism might be in order. Some time ago the present author [Ann. of Math. (2) **48** (1947), 930–948; MR **9**, 241] made an axiomatic study of the foundations of general quantum mechanics which led him to propose such a modification. In the paper under review he explores the possibility of fitting a convergent field theory into this more general framework.

His starting point is the normed Jordan algebra J of all bounded observables. Instead of making the conventional assumption that this Jordan algebra is isomorphic to the algebra of all self adjoint operators in the ring $B(H)$ of all bounded operators in some Hilbert space H, he makes a weaker assumption and allows $B(H)$ to be replaced by any self adjoint subalgebra A which is closed in the normed topology. Moreover, A is supposed to be given only as an abstract C^* algebra and not realized in any particular way as a subalgebra of $B(H)$. An important consequence is that an automorphism of J need not be of the form $T \to UTU^{-1}$, where U is a unitary member of A. This means that the unitary operators of the conventional theory must be replaced by suitable automorphisms of A. Moreover, it suggests that at least some divergences may be attributed to making (in effect) the unjustified assumption that certain automorphisms are implementable by unitary operators.

Assuming that a state is uniquely determined once one knows the expected value of every bounded observable in that state, one can identify the state with this expected value assignment and hence with a certain linear functional f on A. This linear functional must of course have the property that $f(aa^*) \geqq 0$ for all a in A in order that squares of observables should have non-negative expected values. While it is not clear that every linear functional f such that $f(aa^*) \geqq 0$ should be associated with a physical state, the author finds it convenient to define state as though this were the case. In other words, a state is a linear functional f on A such that $f(aa^*) \geqq 0$. By well-known constructions in the theory of Banach * algebras every state f of A defines a concrete representation $a \to V_a$ of A by bounded linear operators in some Hilbert space $H(V)$ in such a manner that $V_a{}^* = (V_a)^*$ and $f(a) \equiv (V_a(\phi), \phi)$ for some unit vector ϕ in $H(V)$. Thus each state can be represented by a Hilbert space vector as

in the conventional theory but the Hilbert space may change from state to state.

In fitting field theory into this framework the author begins with a pair X, X' of real vector spaces and a non-singular bilinear functional B defined on the set of all pairs x, x' with $x \in X$ and $x' \in X'$. The direct sum $X \oplus X'$ is to be regarded as the phase space of the classical dynamical system (field) which is being quantized. X and X' correspond to configuration and momentum space and B to the form $\sum dp_i dq_j$. In the author's terminology such a triple X, X', B is a "single particle structure". His first goal, achieved in section 1 of the paper, is to find the appropriate C^* algebra A for each X, X', B; in other words to find the right quantum mechanical phase space to associate with each classical phase space of the indicated form. This is done via what are called "canonical systems", which are essentially solutions of the Heisenberg commutation relations for the classical phase space in question (only boson fields are considered in this paper). Specifically, a canonical system (over X, X', B) is a pair of mappings $x \to p(x)$; $x' \to q(x')$ defined for all x in X and all x' in X' and having the following properties: (a) for all x and x', $p(x)$ and $q(x')$ are self adjoint operators in a Hilbert space K; (b) $x \to \exp(ip(x))$ and $x' \to \exp(iq(x'))$ are unitary representations of the additive groups of X and X' respectively which are continuous on each finite dimensional subspace; (c) $\exp(ip(x)) \exp(iq(x')) = \exp(iB(x, x')) \times \exp(iq(x')) \exp(ip(x))$.

Let q, p be a canonical system. For each pair M, M' of finite dimensional subspaces of X and X' respectively (which have the property that B restricted to $M \oplus M'$ is non-singular), form the weak closure $A_{M,M'}$ of the ring generated by the $\exp(ip(x))$ and $\exp(iq(x'))$ as x and x' vary over M and M' respectively, and let A denote the uniform closure of the ring generated by all $A_{M,M'}$. According to theorem 1, the ring A, as an abstract C^* algebra, is uniquely determined by X, X', B and is independent of the particular canonical system q, p used in its construction. The author takes A as the C^* algebra which defines the quantum phase space of the system whose classical phase space is $X \oplus X'$. When X and X' are finite dimensional, A turns out to be the set of all bounded linear operators on a Hilbert space, but this is not so in general. The author gives physical arguments to justify the double limiting process employed in defining A.

Having described the algebra of all bounded observables, there remain the problem of describing the dynamics—or at least a substitute for the scattering operator—and the problem of "labeling the states". By the latter is meant finding a substitute for the conventional description of the most general state as a superposition of states having definite values for a complete commuting set of observables. The states of the author are abstract linear functionals and it is not immediately obvious how to relate them to the possible values of observables.

As far as the scattering operator is concerned, it follows from general principles that its role must be played by an automorphism S of A. However the author makes no attempt to state what S should be in any concrete case. His goal is rather to discuss the general statements that can be made assuming that S is at hand. He does not claim to have produced a divergence free field theory—only possible foundations for one.

The author's proposal for labeling the states is based upon a partial return to the conventional formalism via the notion of "vacuum state". As noted above, any particular state defines a concrete realization of A as an algebra of operators on a Hilbert space, and as a canonical such realization it is natural to take the one defined by the state which describes the physical vacuum. However it is by no means obvious, a priori, which state this is. The author proposes to define the physical vacuum state as the unique state which is invariant under S and the members of a certain group of automorphisms of A. This group is that induced by a certain subgroup G_0 of the (assumed) fundamental covariance group of the system. However he is only partially successful in showing that such a definition is possible. Uniqueness is left open, and while the existence of a suitably invariant state is proved under quite general hypotheses (theorem 4), it is not shown that such a state exists having the very desirable property of regularity. (It is only the regular states which define representations of the commutation relations.)

Let A be concretely realized via a suitable vacuum state. Then each unit vector in the corresponding Hilbert space K defines a state, and these states may be expected to play a special role. At any rate they are the ones that the author labels. If G_0 contains translations in space time, then momentum operators may be defined in K in the usual way as the infinitesimal generators of one parameter subgroups. These are far from forming a complete commuting family, however, and in order to have labels corresponding to the conventional ones one needs some analogue of the notion of the number of particles in a given momentum state.

The author proposes to fill this need by exploiting the following observations. Let P be a linear operator in $X \oplus X'$ such that $P^3 = -P$. For each real t let $F(t) = I + \sin(t)P + (1 - \cos(t))P^2$. Then $t \to F(t)$ is a one parameter group of non-singular linear transformations in $X \oplus X'$ which is the identity whenever t is an integral multiple of 2π. When each $F(t)$ preserves the fundamental form B and the vacuum state satisfies certain conditions with respect to the induced automorphisms $\theta(F(t))$ of A, this group of automorphisms induces a one parameter group of operators in K whose infinitesimal generator has integral eigenvalues. The vectors in the n eigenspace are to be thought of as defining states in which there are n particles all in "one particle states" defined by the range of P. One of the conditions on the vacuum state is a sort of approximate invariance under the $\theta(F(t))$. When it is actually invariant, the eigenspaces are mutually perpendicular, but need not be otherwise. In the important special case in which $X \oplus X'$ has the structure of a complex Hilbert space, each P is i times the projection on the appropriate subspace of $X \oplus X'$. How P is to be defined in more general cases is left vague.

We have spoken freely of the automorphisms of A induced by linear transformations in $X \oplus X'$ and of linear transformations on K induced by automorphisms of A. Theorems 2 and 3 spell out in detail how these automorphisms and linear transformations are related. In dealing with K the author does not confine himself to the K's defined by vacuum states but considers arbitrary states of A.

In addition to the proofs of theorems 1–4 and certain corollaries, the paper contains numerous explanatory remarks and a discussion of a number of examples of systems $X, X'B$ with a given covariance group.

To summarize briefly, the author proposes that the

problem created by the existence of many inequivalent solutions of the commutation relations be resolved by seeking the S operator amongst the automorphisms of A and then choosing that particular solution of the commutation relations defined by the vacuum state associated with S and the relevant covariance group. He discusses at length and partially solves some of the technical problems involved in implementing this program.

G. W. Mackey (Cambridge, Mass.)

3478:

Nagy, K. L. Probabilistically interpretable field theories with an indefinite metric. Acta Phys. Acad. Sci. Hungar. **11**, 193–199 (1960). (Russian summary)

In recent years some effort has been put into the investigation of field theories with an indefinite metric in the Hilbert space spanned by the state vectors. The interest in such theories comes partly from the fact that an indefinite metric is a convenient way to describe conventional electrodynamics [S. N. Gupta, Proc. Phys. Soc. Sect. A **63** (1950), 681–691; MR **12** 67; K. Bleuler, Helv. Phys. Acta **23** (1950), 567–586; MR **12**, 465] and partly because of some recent attempts to make a theory of elementary particles with the aid of such a Hilbert space [cf., e.g., W. Heisenberg, Rev. Mod. Phys. **29** (1957), 269–278; MR **19**, 813]. In all these theories one of the main problems is to achieve conservation of probability for those states which have positive norm only in the Hilbert space. Consider a state vector $|t\rangle$ which, for $t=-\infty$, lies entirely in that subspace of the Hilbert space which is spanned by vectors with positive norm. At a later time t this state can be written as $|t\rangle = U(t)|-\infty\rangle$ and must, in general, be supposed to have moved away from the initial, physical subspace. Introduce projection operators P_p for the physical subspace and P_n for the subspace spanned by state vectors with negative norm or zero norm. The vector $P_p|t\rangle = |t\rangle_p$ must therefore be expected to have a norm which is different from 1. The author writes a formal equation of motion for this physical state vector as

$$i\frac{d|t\rangle_p}{dt} = [P_pH(t)P_p + W(t)]|t\rangle_p,$$

$$W(t) = P_pH(t)P_nU(t)P_p(P_pU(t)P_p)^{-1},$$

where $H(t)$ is the interaction Hamiltonian and where the operator $(P_pU(t)P_p)^{-1}$ is determined by means of iteration from the equation

$$(P_pUP_p)^{-1} = P_pU^+P_p - P_pU^+P_nUP_nU^+P_p + P_pU^+P_nUP_p(P_pUP_p)^{-1}P_pUP_nU^+P_p.$$

This last equation can be obtained formally from the requirement that U is a unitary operator and the identity $P_pUP_p(P_pUP_p)^{-1} = P_p$. Obviously, the operator inside the square bracket in the equation of motion above is not an hermitian operator and this causes the non-conservation of probability for the state $|t\rangle_p$ alone. The author now suggests that one should modify this equation of motion so as to make the operator inside the square bracket hermitian simply by adding the hermitian conjugate of the last term. In formal language, this means that one uses

$$i\frac{d|t\rangle_p}{dt} = [P_pH(t)P_p + W(t) + W^+(t)]|t\rangle_p.$$

In this way, conservation of probability is automatically achieved. However, the author himself stresses that the S-matrix obtained in this way does not fulfill any reasonable causality condition. He summarizes his own discussion by saying that even if some physical requirements can be fulfilled in a theory where the Hilbert space has an indefinite metric, "nevertheless a fully acceptable general procedure does not exist".

G. Källén (Lund)

3479:

Kar, K. C.; Purkayastha, Sabita. Linear relativistic Hamiltonian and the electromagnetic field. Indian J. Theoret. Phys. **7** (1959), 25–32.

Author's summary: "The direct method and the method of taking wavestatistical average, for the linearisation of the relativistic Hamiltonian, without matrices are critically discussed. It is shown that the relativistic Hamiltonian is a world vector in the space-time continuum. The linear Hamiltonian without matrices for charged particle in the electromagnetic field is also studied. It is shown that the well-known term involving H representing the energy of a magnetic dipole comes out on recombining the two linear Hamiltonians in the electromagnetic field."

3480:

Fáy, Gy.; Fényes, I.; Tőrös, R. Über die quantenmechanisch Möglichen physikalischen Zustände. Acta Phys. Acad. Sci. Hungar. **11**, 109–115 (1960). (Russian summary)

The authors emphasize that it is necessary to define very carefully the domain of existence of one's operators if one wants to make a rigorous mathematical discussion of operators in a Hilbert space. They show with examples from elementary quantum mechanics that one very easily runs into formal contradictions even with the operators p and q if one is somewhat careless about the states to which these operators are applied.

G. Källén (Lund)

3481:

Grossmann, A. Description of the extended tube. J. Mathematical Phys. **1** (1960), 85–86.

According to A. Wightman [Phys. Rev. (2) **101** (1956), 860–866; MR **18**, 781] one defines the extended tube of n complex four vectors $z_1, \cdots, z_n$ as the manifold where the imaginary parts of these four vectors either lie in the forward light cone or can be brought into the forward light cone with the aid of a complex Lorentz transformation. The purpose of the paper reviewed here is to give an alternative characterization of the extended tube. This is done in the following way. Every complex four vector $c_\mu = a_\mu + ib_\mu$ in a natural way defines a real eight-dimensional euclidean space $E^{(8)}$ with points $C = (a_0, a_1, a_2, a_3; b_0, b_1, b_2, b_3)$. From a given vector $C \in E^{(8)}$ the author defines four other vectors from

$$H^{(0)} = (b_0, b_1, b_2, b_3; a_0, a_1, a_2, a_3),$$
$$H^{(1)} = (b_1, b_0, -a_3, a_2; a_1, a_0, b_3, -b_2),$$
$$H^{(2)} = (b_2, a_3, b_0, -a_1; a_2, -b_3, a_0, b_1),$$
$$H^{(3)} = (b_3, -a_2, a_1, b_0; a_3, b_2, -b_1, a_0).$$

These new vectors H fulfill

$$H^{(i)}\cdot H^{(k)} = |C|^2\delta_{ik} \quad (i, k = 1, 2, 3),$$
$$H^{(0)}\cdot H^{(i)} = 2[a_0\bar{a} + b_0\bar{b} + (\bar{a}\times\bar{b})]_i,$$

where the scalar products and the norm are taken in the euclidean space $E^{(8)}$. The author further defines the "cone associated to C" as the set of vectors $Z \in E^{(8)}$ such that

$$Z\cdot H^{(0)}-[\sum_{i=1}^{3}(Z\cdot H^{(i)})^2]^{1/2}>0.$$

Finally, a cone associated to a complex vector c, the Lorentz square of which is real and time-like (i.e., $c_\mu c_\mu = c_0^2-\bar{c}^2>0$), is called a "distinguished cone". In terms of these definitions the author's main result can be formulated in the following way: An n-tuple $z_1, \cdots, z_n$ of complex four vectors belongs to the extended tube if and only if the convex body spanned by the real eight-dimensional vectors $Z_1, \cdots, Z_n$ lies in the interior of some distinguished cone. *G. Källén* (Lund)

3482:

Terasawa, Tokuo. Spin-orbit splitting and tensor force. I. Progr. Theoret. Phys. **23** (1960), 87–105.

Author's summary: "The effect of the tensor force on the spin-orbit splitting in He^5 and N^{15} is examined by using the meson-theoretic potential and the phenomenological Serber potential which are consistent with the experimental data of two nucleon systems. About a half of the experimental values of the spin-orbit splitting in the mentioned nuclei are obtained by the accurate computation of the second order effect in perturbation theory, whereas several previous calculations of this effect have yielded the splitting of wrong sign or of too small magnitude.

"As is pointed out in the present computation, the deformation of the closed shell core induced by the tensor interaction between the nucleons in the core is restricted so as to satisfy the Pauli principle with the outside nucleon. This restriction is mainly responsible for the present result of splitting energy."

3483:

Arima, Akito; Terasawa, Tokuo. Spin-orbit splitting and tensor force. II. Progr. Theoret. Phys. **23** (1960), 115–136.

Authors' summary: "General formulae of the second order perturbation energies due to the tensor force are given in the case of the closed shell + one nuclei, and useful formulae for calculating the two-body matrix elements are also derived. Using these formulae, the D-state doublet splitting in O^{17} is estimated and it is found that about a half of the observed value is explained in terms of the second order effect of the tensor force as in the case of He^5 and N^{15}."

3484:

Bródy, T. A. Transformation brackets for functions of an harmonic oscillator. Tables for $n_1=n_2=0$. Rev. Mexicana Fís. **8** (1959), 139–227. (Spanish. English summary)

An eight-figure table is given of the brackets $\langle nl, NL, \lambda | 0l_1, 0l_2, \lambda\rangle$ for $l_1 \geqq 6$, $l_2 \geqq 6$, $l \geqq 6$, $L \geqq 12$, where $\langle nl, NL, \lambda | n_1 l_1, n_2 l_2, \lambda\rangle$ is a transformation coefficient relating a two-nucleon wave function (n_1, n_2, l_1, l_2 being the radial and angular momentum quantum numbers of the two particles moving in the common harmonic oscillator potential) to the centre-of-mass-relative-coordinate wave function (N, L, n, l being the corresponding quantum numbers for the centre-of-mass and the relative motion); λ is the total angular momentum. *D. ter Haar* (Oxford)

3485:

Vanagas, V. V. On calculation of the matrix elements of operators depending on the coordinates of several particles. Trudy Akad. Nauk Litov. SSR. Ser. B **3 (15)** (1958), 3–16. (Russian. Lithuanian summary)

In approximation methods for solving the Schrödinger equation for many-electron atoms, the wave function can be expressed conveniently as a function of the relative coordinates of the electrons. The calculation of matrix elements of operators dependent on the coordinates (or momenta) of the particles then presents difficulties. The author discusses reduction formulas for such matrix elements, making use of the properties of the spherical harmonics and the vector coupling model.

E. L. Hill (Minneapolis, Minn.)

3486:

John, T. L. The numerical solution of the exchange equations for slow electron collisions with hydrogen atoms. Proc. Phys. Soc. **76** (1960), 532–538.

Author's summary: "Numerical methods are used to calculate phases in the exchange approximation for the s-, p- and d-wave scattering of electrons by hydrogen atoms, using small energy intervals. There is a discussion of experimental results."

3487:

Breene, R. G., Jr. Analytic wave functions. I. Atoms with 1s, 2s, and 2p electrons. Phys. Rev. (2) **111** (1958), 1111–1113.

Analytic wave functions containing certain parameters are obtained by minimizing the energy expression in terms of those parameters. The computer program described is applicable to atoms with 1s, 2s and 2p electrons, and results are reported for the ground-state configurations of O and O^-. *D. F. Mayers* (Oxford)

3488:

Breene, R. G., Jr. Analytic wave functions. II. Atoms with 1s, 2s, 2p, 3s, and 3p electrons. Phys. Rev. (2) **113** (1959), 809–813.

Describes an extension of the program described in the previous paper to deal with 3s and 3p electrons. Results are given for 12 configurations of aluminium and 14 of oxygen. *D. F. Mayers* (Oxford)

3489:

Breene, R. G., Jr.; Nardone, Maria C. Wave function for the free electron. I. The Coulomb potential. Phys. Rev. (2) **115** (1959), 93–96.

An expression is derived for the potential function due to the Coulomb forces of the nucleus and the core electrons, using analytic wave functions for these electrons. The

Schrödinger equation for the free electron is then solved numerically in this potential. s and d wave functions for an electron in the presence of an oxygen atom are tabulated.
D. F. Mayers (Oxford)

3490:

McWeeny, R. Hartree-Fock theory with nonorthogonal basis functions. Phys. Rev. (2) **114** (1959), 1528–1529.

The variational solution of the many-particle problem in terms of a linear combination of basic functions may be obtained by iterative construction of the density matrix. It is shown that the method may be adapted for non-orthogonal basic functions, avoiding the cumbersome construction of an orthogonal set.
D. F. Mayers (Oxford)

3491:

Flodmark, Stig. Note on a BESK programme for molecular one-electron two-center integrals and some diagrams showing the dependence of these integrals on the orbital exponents. Ark. Fys. **17**, 81–88 (1960).

A description of a computer programme for calculating the integrals involving K, L and M atomic orbitals. Graphs are given of some of these integrals as functions of the inter-atomic distance. *D. F. Mayers* (Oxford)

3492:

Tiberio, Ugo. Sulle caratteristiche elettromagnetiche dello spazio nei campi nucleari ed atomici. Ricerca Sci. **30** (1960), 553–560. (French, English and German summaries)

Author's summary: "On the ground of the values of ε and μ of empty space, according to Einstein-Schwarzschild transformation, which have been deduced in a former work by the author, an attempt is made to calculate the electromagnetic radius of the nucleus of heavy elements and of the atom. To this end, certain features are attributed to the mass nuclear field and to the atom centripetal field, which are analogous, from the relativistic point of view, to those of the gravitational field. The results of this calculation do not seem to be in contrast with the experimental values.

"Furthermore, other deductions too, concerning nuclear spaces in connection with the dimensions of the elementary charge, e.g., the non-existence of elementary separated magnetic mass, the velocity of electromagnetic perturbations and the mass excess, are not in contrast with the nuclear patterns which are most frequently adopted. The author concludes pointing out the advisability of carrying out further researches concerning the limits and the relations between general classical relativity and modern physics."

3493:

Pratt, G. W., Jr. Generalization of band theory to include self-energy corrections. Phys. Rev. (2) **118** (1960), 462–467.

Author's summary: "A one-particle Schrödinger-like equation is found whose eigenvalues in certain cases are identical with the energies of the many electron states of a semiconductor or insulator including self energy corrections. The one particle Hamiltonian is expressed in terms of the Coulomb interaction as modified by polarization processes. The relation is given between the modified Coulomb interaction and the dielectric function which is the generalization of the classical dielectric constant. Suggestions are made as to how the one-particle equation including self-energy effects might be solved in practice."

3494:

Paul, David I. Theory of magnetism and the ground-state energy of a linear chain. Phys. Rev. (2) **118** (1960), 92–99.

Author's summary: "The theory of strong magnetic effects is investigated from the point of view of orthogonal atomic functions for the case of one dimension. Thus, the exchange integral is considered positive and the interaction between the polar and the nonpolar states for all possible arrangements of electron spins is included in our formulation of the problem. The resulting secular equations are solved for both large and small interactions between states for the case of only one electron spin oriented in a direction opposite to all other electron spins, and they are solved for small interactions between states for the more general case of any number of electron spins being in a given direction. It is shown how inclusion of the polar states can yield either a ferromagnetic or an antiferromagnetic ground state depending on the difference in absolute magnitude between the exchange integral and the sum of other integrals representing electron-nuclei interactions."

3495:

Goldfarb, L. J. B.; Johnson, R. C. Angular distributions and polarization in stripping processes and in direct reactions. Nuclear Phys. 18 (1960), 353–394.

Authors' summary: "General expressions are found for angular distributions and polarization effects in a variety of direct reactions. These reactions include processes involving the stripping of deuterons and of heavier nuclei, in addition to inelastic-scattering phenomena which arise either from individual nucleon-nucleon encounters or from collective effects at the nuclear surface. Various selection rules are found for the existence of polarization, using the distorted-wave Born approximation and allowing for spin-dependent distortions. The influence of a neighbouring isolated resonance state of the compound nucleus is examined, and modifications are found of the general results. Expressions are given for the angular correlation of the γ-rays following de-excitation of the residual nucleus."

3496:

Mehta, M. L. On the statistical properties of the level-spacings in nuclear spectra. Nuclear Phys. **18** (1960), 395–419.

Author's summary: "The level-spacing distribution is studied under the random-matrix hypothesis. Rigorous lower and upper bounds for the distribution function of the level-spacing are given. Comparison of these with Wigner's surmise shows that it is a very good approximation. It is shown, however, that it cannot be exact even in the limiting case $n\to\infty$."

3497:

Grishin, V. G.; Ogievetski, V. I. On the minimum number of partial waves in two-body reactions. Nuclear Phys. **18** (1960), 516–520.

Authors' summary: "A reliable method is offered to determine the minimum number of partial waves involved in the interaction, if the total elastic cross section and differential cross section at a given angle are known."

3498:

Lyness, J. N. A method for calculating scattering phase shifts. Nuclear Phys. **18** (1960), 654–668.

Author's summary: "A method, based on perturbation theory, is derived for obtaining the reflection coefficient for scattering a plane wave by a potential well of a particular type. This type is an extension of a simple potential well, whose reflection coefficient is known. This method has the advantage that exact limits can be placed on the possible error due to using the perturbation expansion. A possible application is the exact determination of the effect on the cross section of a 'tail' added to a potential well."

3499:

Smith, Robert C.; Sharp, Robert T. Central three-body forces in heavy nuclei. Canad. J. Phys. **38** (1960), 1154–1167.

Authors' summary: "Three-nucleon potentials are calculated using the renormalized, static theory of Chew and Low. These potentials are used to evaluate the three-body contribution to the total energy per nucleon in nuclear matter as a function of nuclear density. It is found that while the three-body energy is greater than that predicted by unrenormalized theories by about one order of magnitude as a result of multiple-scattering effects, its dependence on the nuclear density in the region of the equilibrium density is very weak. Three-body forces are therefore not expected to change the saturation properties of nuclear matter as predicted by a hard core potential to any appreciable extent."

3500:

Ivanenko, I. P. On the theory of the passage of the nuclear cascade through the atmosphere. Ž. Èksper. Teoret. Fiz. **37** (1959), 1046–1049 (Russian); translated as Soviet Physics. JETP **10** (1960), 744–746.

The author gives the solution of the integro-differential equations describing the growth of a nuclear cascade in the atmosphere. Rather than giving the usual solution with the boundary condition taken at zero depth, a series solution in the form of 'successive generations' is given with the boundary condition valid for an arbitrary depth. *H. Messel* (Sydney)

3501:

Satchler, G. R. Symmetry properties of the distorted wave theory of direct nuclear reactions. Nuclear Phys. **18** (1960), 110–121.

Cross sections for "direct" reactions, such as deuteron stripping, have been understood qualitatively since the work of S. T. Butler. However, corrections to the Butler formula, to take into account the optical model potentials acting on the initial and emitted particles, involve complicated numerical calculations. A semi-classical approach due to Austern and Butler allows understanding of these corrections. In this paper, certain results of the semi-classical approach (e.g., the opposing effects of incoming- and outgoing-wave-distortions on the polarization) are derived directly from the quantum-mechanical distorted wave formula, by means of a symmetry equation (rotation about $\mathbf{q}=\mathbf{k}_1-\mathbf{k}_2$ through 180°) which interchanges the distortion of initial and final waves, but is not equivalent to straightforward time reversal. The symmetry is usually only approximate, not exact.

J. M. Blatt (Sydney)

3502:

Sahni, R. C.; LaBudde, C. D. Study of molecular integrals. I. Two-center exchange integrals. J. Chem. Phys. **33** (1960), 1015–1021.

Authors' summary: "The evaluation of the two-center two-electron exchange integrals using prolate spheroidal coordinates and spherical coordinates is discussed. Estimates for the convergence rates of the infinite series which arise when both coordinate systems are used to evaluate the integrals are obtained."

3503:

LaBudde, C. D.; Sahni, R. C. Study of molecular integrals. II. Three-center one-electron and two-electron integrals. J. Chem. Phys. **33** (1960), 1022–1027.

Authors' summary: "The evaluation of the three-center one-electron and two-electron integrals using spherical coordinates is discussed. Estimates for the convergence rates of the infinite series which arise in all cases are derived."

3504:

Preuss, H. ★Integraltafeln zur Quantenchemie. Bd. 4. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960. viii+145 pp. DM 45.00.

This volume maintains the high standard set by earlier members of the series. It contains a general discussion of the problem of many-centred integrals for both exponential and Gaussian functions, and the use of these functions to approximate numerical Hartree-Fock functions. The tables of integrals and auxiliary functions are accompanied by an extensive survey of existing literature.

A. C. Hurley (Melbourne)

3505:

Hirschfelder, Joseph O. Mathematical bottlenecks in theoretical chemistry. Frontiers of numerical mathematics, pp. 83–97. University of Wisconsin Press, Madison, Wis., 1960.

Three mathematical problems which block the development of wide areas of theoretical chemistry are discussed. These are the following. I. Finding satisfactory approximate solutions of the electronic wave equation for molecular systems. II. The development of a generalized Boltzmann equation for dense gases of molecules possessing internal degrees of freedom. III. Formulating the theory of chemical reaction rates in a form suitable for obtaining practical solutions. The first of these problems

has already been reduced to one of computational difficulty, and with the widespread introduction of large digital computors we can expect rapid progress. Although a similar reduction of the other two problems is not yet possible, computers also play an important role in these fields by supplementing laboratory experimentation with mathematical experimentation provided by Monte Carlo calculations and the like. *A. C. Hurley* (Melbourne)

3506:

Fraga, Serafin; Mulliken, Robert S. Role of Coulomb energy in the valence-bond theory. Rev. Mod. Phys. **32** (1960), 254–265.

The Coulomb part of the binding energy in the valence bond method is calculated for the ground and excited states of H_2 and the ground states of LiH, BH, CH, NH, HF, Li_2, N_2 and F_2 for various degrees of hybridization of the atomic orbitals. The Coulomb energy is much larger for $p\sigma$ and especially $s-p\sigma$ hybrid bonds than for s bonds. In typical cases it amounts to 30–40% of the gross bond energy as compared with 1% for H_2. For Li_2 the figure is 93%. *A. C. Hurley* (Melbourne)

3507:

Allen, Leland C. Hartree-Fock equations with a perturbing field. Phys. Rev. (2) **118** (1960), 167–175.

Author's summary: "The Hartree-Fock equations under the action of an arbitrary field for any order of perturbation are set up in an integro-differential form. This form appears particularly advantageous for practical computation in such problems as electronic polarizability and electronic structure perturbations caused by nuclear moments. The equations are explicitly written down for a uniform perturbing field and a comparison is made with previous formulations. A wide variety of other applications is also discussed." *A. C. Hurley* (Melbourne)

3508:

Mulliken, R. S. Self-consistent field atomic and molecular orbitals and their approximations as linear combinations of Slater-type orbitals. Rev. Mod. Phys. **32** (1960), 232–238.

Until recently almost all approximations to Hartree-Fock functions for molecules were expressed as linear combinations of valence state atomic orbitals each of which was approximated by a single Slater function. Here a systematic notation is developed for more accurate calculations in which a wider variety of more accurate atomic orbitals is employed. *A. C. Hurley* (Melbourne)

3509:

Löwdin, Per-Olov. Expansion theorems for the total wave function and extended Hartree-Fock schemes. Rev. Mod. Phys. **32** (1960), 328–334.

It is shown that the convergence of the superposition of configurations expansion for many electron systems can be optimized by transforming the basis to that of natural spin orbitals which diagonalize the first order density matrix. Generalizations of the Hartree-Fock scheme by the use of projection operators and/or correlation factors are considered. The accuracies of the resulting schemes are estimated by detailed calculations on the helium atom. *A. C. Hurley* (Melbourne)

3510:

Thorson, Walter R.; Nakagawa, Ichiro. Dynamics of the quasi-linear molecule. J. Chem. Phys. **33** (1960), 994–1004.

Author's summary: "Calculations of energy levels, eigenfunctions, dipole transition moments, and other significant properties have been made for a phenomenological model of a two-dimensional harmonic oscillator with a superposed barrier at the minimum of the harmonic potential. A high-speed electronic digital computer was employed. The method of calculation does not involve perturbation theory and converges rapidly for all barrier heights. The problem concerns the behavior of the degenerate bending vibration of a linear triatomic molecule, for example, when a potential barrier is introduced at the linear configuration. In the limit of a high barrier, the motion is that of a 'bent' molecule with rotational motion superposed on the bending vibration about the minimum of potential energy. Results for moderate to low barriers, and those for anharmonic vibration in a well with no barrier, show that the rigid classification of molecules as 'linear' or 'bent' is not possible in some cases and that the intermediate case of the 'quasi-linear' molecule must be considered. The dynamical behavior of such anharmonic motion and its interaction with other vibrations have been investigated, and the primary identifying characteristics of the quasi-linear molecule delineated.

"The analytical method of solution used has also been applied successfully to the one-dimensional barrier problem."

3511:

Parr, Robert G. Three remarks on molecular orbital theory of complex molecules. J. Chem. Phys. **33** (1960), 1184–1199.

Author's summary: "Three suggestions are made and discussed concerning the generalized Hückel and related methods for treating the quantum chemistry of complex unsaturated molecules: (1) Justification for the assumptions of zero overlap and zero differential overlap resides in two facts: (a) For many molecules the atomic orbitals in the LCAO molecular orbitals may be replaced by the corresponding orthogonalized atomic orbitals of Löwdin, without effect on the molecular orbitals. (b) Integrals involving charge distributions which are products of orthogonalized atomic orbitals are small. (Observations previously made by several authors.) (2) Molecules isoelectronic with benzene are conveniently handled making use of molecular orbitals appropriate to the full benzene symmetry, for then deviations from benzene symmetry can be classified and treated systematically using symmetry combinations of basic integrals. (Formulas are given.) (3) The two-center coulomb repulsion integrals, previously dealt with by rather arbitrary semiempirical procedures, may be expanded by multipole expansion methods, leaving the independent multipoles of each atomic orbital as its defining theoretical or semiempirical characteristics."

3512:
Stahl-Brada, R.; Low, W. Tables of eigenvalues and matrix elements of transition probabilities for an axial spin Hamiltonian with $S=3/2$. Nuovo Cimento (10) **15** (1960), supplemento, 290–334.

3513:
Baym, Gordon. Inconsistency of cubic boson-boson interactions. Phys. Rev. (2) **117** (1960), 886–888.

Author's summary: "It is shown that there does not exist a ground state for a system of spin zero bose fields coupled only by local interactions involving three powers of the fields. Thus these interactions alone are not suitable for a model of interacting fields."

3514:
de Oyarzabal, Juan. Angular distributions in beta decay. Rev. Mexicana Fis. **9** (1960), 1–33. (Spanish. English summary)

Author's summary: "A general expression is obtained for the probability of the 'allowed' transitions in beta decay, taking account not only of coulomb corrections but also of the possibility of a non-vanishing mass of the neutrino. In section III the meaning of each term of this expression is analyzed leading to particular results by summing over the spins and directions of emission of the different particles emitted. In section IV the behaviour of each term of the expression with respect to conservation of parity, time reversal and charge conjugation is analyzed. Finally, in section V the effect of the non-zero mass of the neutrino on the angular distributions and the form of the energy spectrum of the electron is considered."

3515:
Jackson, J. D. ★The physics of elementary particles. Investigations in Physics, Vol. 9. Princeton University Press, Princeton, N.J., 1958. x+135 pp. $4.50.

This useful little book attempts to present an introductory account of elementary particle physics with a minimum of formal apparatus.

More than a third of the book is concerned with what one may call "classical" pion-physics. The treatment is based on the static model and uses semi-classical arguments, but there is also a short discussion of the proper theoretical treatment of scattering, including the essentials of dispersion relations. The section on the photoproduction of pions is particularly neat and enjoyable. The second part of the book gives a concise account on the phenomenologics of hyperons and K-mesons, while the remaining 40 pages give a brief discussion of conservation laws in general and a somewhat more detailed review of weak decay processes.

In general, the book is easy to read and abundant references facilitate the interested reader's finding more details in the relevant original publications. One drawback of the work is its slightly unsystematic way of presentation and the great number of ad hoc types of argument and proof. Probably this is the consequence of the fact that the book originated from a series of lectures at a time when progress in the field was so fast that the author could not find sufficient time to follow the classical rule: "Novum in annos prematur".

P. Roman (Boston, Mass.)

3516:
Peaslee, D. C. Seven-dimensional charge space. Phys. Rev. (2) **117** (1960), 873–886.

A seven-dimensional internal space is assumed for the classification of all elementary particles (except photons and gravitons) and their interactions. The baryons form an eight-component spinor, the bosons a seven-component vector, and the leptons an eight-component spinor where no particle-antiparticle distinction is possible. The main conclusions of the scheme are: (i) lepton conservation must be abandoned, but this automatically introduces parity-nonconservation into β-decay; (ii) the boson-fermion interaction is predominantly pseudovector; (iii) the Σ-Λ mass-difference arises from a two boson-two fermion interaction, and the N-Ξ mass difference has an "intrinsic" origin. Further results concern the $\Delta S=\pm 1$ and $\Delta I=\pm\frac{1}{2}$ rules. *P. Roman* (Boston, Mass.)

3517:
Kretzschmar, Martin. Statistische Gewichte für ein System vieler Teilchen mit beliebigen Spins. Z. Physik **157** (1960), 554–557. (English summary)

In the theory of a large number of particles of given spins (or isotopic spins), it frequently is necessary to estimate the number of states available for a given total spin (or isospin). The quantity in question is obtainable from formulas for the reduction of an n-fold Kronecker product of representations of the 3-dimensional rotation group, with final summation over those irreducible representations of given total spin. The author makes use of group-theoretical methods for the derivation of a general formula for this number.

E. L. Hill (Minneapolis, Minn.)

3518:
Yamaguchi, Yoshio. A composite theory of elementary particles. Progr. Theoret. Phys. Suppl. No. 11 (1959), 1–36.

As a special case of the Sakata-model, a theory of composite particles is developed. The basic particles are the p, n, Λ on one hand and the ν, e, μ on the other. Besides the electromagnetic coupling, very strong, moderately strong and weak interactions are assumed. The very strong interactions are assumed to act only on the baryons, are charge independent and are completely symmetrical with respect to p, n and Δ. They give rise to the major parts of the observed baryonic mass, and they are also responsible for the creation of various bound states such as pions, kaons, etc., which are composites of baryon-antibaryon pairs. The moderately strong interactions are also assumed to be charge independent. They split the mass-degeneracy between nucleons and Λ, between pion and kaon, and also between (e, ν) and μ. Specific models of the structure of the hyperons are set up and it is concluded that the kaon is pseudoscalar, and the Λ-Σ and Ξ-N relative parities are odd. The Feynman-Gell-Mann theory of weak interactions is consistently transferred into the present scheme. The paper ends with some speculations of the existence of extremely weak interactions and possible violations of charge and/or baryon conservation.

P. Roman (Boston, Mass.)

3519:

Yamaguchi, Yoshio. A model of strong interactions. Progr. Theoret. Phys. Suppl. No. 11 (1959), 37–51.

The aim of this paper is to construct and examine a local Yukawa theory, the form of which should be the phenomenological representative form to which the composite particle theory of the present author [see preceding review] must lead. First, a "global transformation" is defined as that which gives the effect on the nucleon- and meson-states that arise from the assumed invariance under the permutation $n\leftrightarrow\Lambda$. Then the "global Yukawa interaction" is characterized by charge independence and invariance under the global transformation. This enables one to construct explicitly all relevant global (and renormalizable) Yukawa interactions. They possess a higher symmetry than the usual charge independent interactions. On the other hand, it turns out that the whole theory can work only if there exist some "new particles". In particular an isosinglet, non-strange and pseudoscalar neutral meson is predicted. *P. Roman* (Boston, Mass.)

3520:

Kretzschmar, Martin. Zur Theorie der Wignerschen Supermultipletts. Z. Physik **157** (1960), 558–567.

When neutrons and protons are treated as identical particles, the Pauli principle states that the wave function of a system of nucleons is antisymmetric under a permutation of the parameters of any pair of nucleons. The parameters of a nucleon are its cartesian coordinates, spin, and isotopic spin (isospin). Frequently it is convenient to consider this permutation group as the direct product $P=P_c\times P_S\times P_T$, where P_c is the permutation group on the cartesian coordinates, P_S that on the spins, and P_T that on the isospins. By combination of the latter groups we obtain $P_{ST}=P_S\times P_T$. The representation theory of P_{ST} provides the construction of the Wigner supermultiples. In applications the symmetry of a wave function under P_c must be adapted to its symmetry under P_{ST} in order that the Pauli principle be satisfied.

The author studies the representation theory of P_{ST} by group theoretical methods. Formulas are given for the construction of the supermultiplets with given symmetry properties. {The argumentation is given in very compact form, with references to the literature for notation and the theory of the permutation groups.}

E. L. Hill (Minneapolis, Minn.)

3521:

Engelmann, F. Zur Frage der bei Elementarteilchen möglichen Spins. Nuovo Cimento (10) **14** (1959), 1366–1372. (English and Italian summaries)

Author's summary: "It is shown that it is possible to formulate the phenomenological description of elementary particles in a way which allows the existence of particles of the spins 0, $\frac{1}{2}$ and 1 only. This restriction appears as a consequence of the fact that space-time is 4-dimensional."

3522:

Gupta, Suraj N. Pion theory of nuclear forces with nucleon recoil. Phys. Rev. (2) **117** (1960), 1146–1151.

Author's summary: "The nuclear potential between two nucleons with nonrelativistic velocities in their center-of-mass system is calculated by using the relativistic pion theory and taking fully into account the effect of the nucleon recoil. The resulting potential completely disagrees with the Klein potential but differs from the Lévy potential to a lesser extent. It is shown that an expansion of the contribution of the nucleon recoil in powers of the ratio of the pion and nucleon masses leads to erroneous results."

3523:

Macfarlane, A. J. Unitarity conditions for binary processes involving particles of arbitrary spin. Nuclear Phys. **18** (1960), 570–574.

Author's summary: "Using the helicity description of two-particle eigenstates of total angular momentum, it is found that a concise statement can be obtained of the unitarity condition in the two-particle approximation, for processes of type $A+B\to C+D$ involving particles of arbitrary spin. As an illustration of the use of this statement a specific set of processes is discussed and some remarks are made about its application below the physical threshold."

3524:

Wilson, J. G.; Wouthuysen, S. A. (Editors). ★Progress in elementary particle and cosmic ray physics. Vol. V. Series in Physics. North-Holland Publishing Co., Amsterdam; Interscience Publishers Inc., New York; 1960. xii + 461 pp. $10.75.

A collection of five review articles. Those of mathematical interest will be reviewed separately.

3525:

Emel'yanov, A. A.; Černavskiĭ, D. S. Effect of viscosity in multiple production on the energy distribution of secondary particles. Ž. Èksper. Teoret. Fiz. **37** (1959), 1058–1061 (Russian); translated as Soviet Physics. JETP. **10** (1960), 753–755.

Author's summary: "The effect of viscosity on processes taking place in a simple wave is considered in the hydrodynamical theory of multiple production of particles. It is shown that the effect of viscosity on the energy distribution of the fastest particles may be significant at sufficiently high energies."

3526:

Meijer, Paul H. E. A group theoretical proof of Kramers' theorem. Physica **26** (1960), 61–65.

Kramers' theorem on the degeneracy of an n-electron system is discussed from the point of view of group representation theory instead of time reversal theory. However, an error is made in applying a criterion of Frobenius, and the results are inconclusive.

J. S. Lomont (New York)

3527:

McWeeny, R. Some recent advances in density matrix theory. Rev. Mod. Phys. **32** (1960), 335–369.

This paper is mainly concerned with applications of density matrix (d.m.) theory to the quantum mechanical N-electron system where the system (atom, molecule, crystal) is assumed to be in a definite eigenstate (usually

its ground state). The general d.m. theory is outlined and Husimi's reduced d.m. matrices are introduced—unfortunately not normalized to unity but to N which has led to an unfortunate slip in describing the idempotency for these reduced d.m. which is the condition that the N-electron wave function be a Slater determinant as the condition for a pure state which it is not.

The relation between the two-electron d.m. and electron correlations is discussed. After that an extensive discussion is given of the following topics which are currently of importance in d.m. work done by quantum chemists: orbital approximations (Slater method, Hartree-Fock approximation, configuration interaction), purification of an almost idempotent matrix, electron groups and generalized product functions, generalization of the self-consistent field method, and applications (intermolecular forces, split-orbital wave functions, spin densities).

D. ter Haar (Oxford)

3528:

Fradkin, E. S. The Green's function method in quantum statistics. Nuclear Phys. **12** (1959), 465–484.

The Green's function method, a technique employed in the quantum theory of fields [see for instance J. Schwinger, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 452–459; MR **13**, 520], is adapted in this paper to the investigation of questions of quantum statistics. While in quantum field theory the Green's functions are defined as vacuum expectation values, in quantum statistics they are defined as grand-canonical averages. Two types of Green's functions can be considered corresponding to time dependent second quantized Heisenberg operators or to temperature dependent operators satisfying a Bloch type equation. One can see that these two types of Green's functions can be obtained the one from the other by analytic continuation in the time (temperature) variable. The Green's functions satisfy functional equations analogous to those of quantum field theory. The author discusses a number of special situations and obtains approximate solutions of the functional equations from which a number of well known results of many particles theory can be extracted.

The content of this paper overlaps with that of a number of other papers on the subject which have recently appeared (some of the work seems to be definitely earlier than that of the present author: see for instance P. Martin and J. Schwinger, Phys. Rev. (2) **115** (1959), 1342–1373 [MR **22** #588], where many earlier references can be found). An original feature of the author's paper is its emphasis on relativistic invariance. To this reviewer, however, the interest of this development seems to be of little more than formal significance.

B. Zumino (New York)

3529:

Lee, T. D.; Yang, C. N. Many-body problem in quantum statistical mechanics. V. Degenerate phase in Bose-Einstein condensation. Phys. Rev. (2) **117** (1960), 897–920.

The method developed in a previous paper [Phys. Rev. (2) **117** (1960), 22–36; MR **22** #581] is extended by the introduction of a so-called x-ensemble to which an Ursell-type expansion can be applied. One can then discuss the case of a degenerate system of bosons, that is, a system in which the ground state occupation number is comparable to the total number of particles in the system. This method is then applied to a dilute system of hard sphere bosons. The possible connection with the p-T-diagram of this dilute gas and that of He is discussed.

D. ter Haar (Oxford)

3530:

Levine, Howard B. Diagram expansions in quantum statistics. Phys. Fluids **3** (1960), 225–245.

Author's abstract: "The Montroll-Ward-Lee-Yang approach to quantum statistics is generalized to multicomponent systems. It is also generalized so as to include external fields. The formalism is constructed in a volume-dependent manner and includes internal coordinates, such as spin, from the beginning. It is rigorously proved that the quantum-mechanical volume-dependent cluster integral may be expressed in terms of connected diagrams only. The rules for drawing these diagrams are given. By simply generalizing the meaning of the word 'determinant', all arguments are made to apply to both Fermi-Dirac and Bose-Einstein statistics simultaneously. A statistics factor, $\gamma = \pm 1$, for bosons (fermions) is introduced, in terms of which single formulas apply to both statistics. Rules are stated, by means of which the γ dependence of the contribution to the pressure for any diagram is given in terms of an elementary topological property of the diagram." *A. Klein* (Philadelphia, Pa.)

3531:

Širkov, D. V. On taking the Coulomb effects into account in the theory of superconductivity. Ž. Èksper. Teoret. Fiz. **37** (1959), 179–186 (Russian); translated as Soviet Physics. JETP **10** (1960), 127–131.

The method of the renormalization group in quantum field theory is used to evaluate Coulomb effects in the theory of superconductivity. The results of Bogoliubov, Tolmachev, and Shirkov are recovered, as the lowest order terms in a systematic, and supposedly rapidly convergent, expansion. This is surprising, since the criterion for existence of superconductivity, found this way, involves a "large logarithm" which indicates superconductivity in almost all metals, in qualitative disagreement with experiment. The simple-minded approach, using screened Coulomb forces between electrons, gives much better agreement with experiment.

J. M. Blatt (Sydney)

3532:

Bayman, B. F. A derivation of the pairing-correlation method. Nuclear Phys. **15** (1960), 33–38.

The expectation value of the "reduced Hamiltonian" in the B.C.S.-Bogoliubov theory of superconductivity is evaluated with a trial wave function which is an eigenfunction of the number of particles. By making a saddle-point approximation, one recovers the usual results, but now it is possible to estimate corrections. These are negligible in statistical mechanics, but significant in nuclear physics applications. *J. M. Blatt* (Sydney)

RELATIVITY

See also 3381, 3416, 3560, 3582, 3583.

3533:

Кузнецов, Б. Г. [Kusnecov, B. G.] ★Принцип относительности в античной, классической и квантовой физике. [The principle of relativity in ancient, classical, and quantum physics.] Akademiya Nauk SSSR, Institut Istorii Estestvoznaniya i Tehniki. Izdat. Akad. Nauk SSSR, Moscow, 1959. 232 pp. 5.50 rubles.

A popular account, written in informative style but without mathematical formulas. There are four chapters: isotropy of the world and the concepts of relative and absolute motion in the dynamics of antiquity; homogeneity of space and the classical principle of relativity; homogeneity of space-time and the Einstein theory of relativity; the principle of relativity in quantum physics and the macroscopic homogeneity of discrete space-time.

3534:

Pauli, W. ★Theory of relativity. Translated from the German by G. Field, with supplementary notes by the author. Pergamon Press, New York-London-Paris-Los Angeles, 1958. xiv+241 pp. $6.00.

This is a translation of the article "Relativitätstheorie" in *Encyklopädie der mathematischen Wissenschaften*, vol. V 19, pp. 538–775 [Teubner, Leipzig, 1921], with a number of notes added by the author in order to help bring it up to date. The book consists of five parts: Part I, "The foundations of the special theory of relativity", presents the historical background, the postulates of the theory, and the Lorentz transformation and some of its consequences. Part II, "Mathematical tools", deals with transformations, tensors, and Riemannian geometry in a four-dimensional space-time world. Part III, "Special theory of relativity, further elaborations", consists of (a) kinematics, (b) electrodynamics, (c) mechanics and general dynamics, and (d) thermodynamics and statistical mechanics. Part IV, "General theory of relativity", presents a historical review of earlier work, the postulates of the general theory, deductions from the equivalence principle, the influence of the gravitational field on material phenomena, field equations and variational principle, comparison with observation, some exact solutions of the field equations, the linear approximation, gravitational energy, the Mach principle, and the modification of the equations by the addition of the λ-term. Part V, "Theories on the nature of charged elementary particles", contains a discussion of the electron in special relativity, the theories of Mie, Weyl, and Einstein, and some remarks on the problem of matter.

The book provides a clearly written, critical presentation of the special and general relativity theories, and it includes a large number of references to the early literature in this field. *N. Rosen* (Haifa)

3535:

Jankiewicz, C. Über die Bewegungsgleichungen in linearen Feldtheorien. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 7 (1959), 175–178. (Russian summary, unbound insert)

The Lorentz-invariant equations of motion of charged particles are obtained by varying the usual (generally covariant) action integral with respect to the metric tensor, space-time remaining flat during the variation. This procedure is an alternative to the usual one of varying with respect to the world-lines of the particles. [W. Pauli, *Theory of relativity*, p. 88; see preceding review]. *D. W. Sciama* (London)

3536:

★Colloque sur la théorie de la relativité, tenu à Bruxelles les 19 et 20 juin, 1959. Centre Belge de Recherches Mathématiques. Librairie Universitaire, Louvain; Gauthier-Villars, Paris; 1960. 122 pp.

A collection of 10 articles which will all be reviewed separately.

3537:

Kaplon, M. F.; Yamanouchi, T. A survey of relativistic transformations. Nuovo Cimento (10) 15 (1960), 519–536. (Italian summary)

In obtaining such quantities as thresholds, Q-values, etc., it is usually the practice to go to the center-of-mass or rest-system and then transform back to the lab-system. It is shown that this labor may be conveniently circumvented by appropriate use of invariance properties. General formulae relating to the transformation of distributions from one Lorentz system to another are given and their uses demonstrated. Particular application is made to the γ-ray distribution arising from π^0-decay. *P. Roman* (Boston, Mass.)

3538:

Schwinger, Julian. Euclidean gauge transformation. Phys. Rev. (2) 117 (1960), 1407–1408.

This note discusses gauge transformation within the Lorentz gauge (which is really a class of gauges). In particular, the elimination of the longitudinal field in the author's Euclidean formalism for the Maxwell field [Phys. Rev. (2) 115 (1959), 721–731; MR 22 #2361] is carried out by means of functional methods. It is thereby emphasized that the transverse gauge has a preferred position in the Euclidean framework. *S. Deser* (Waltham, Mass.)

3539:

Das, Anadi J. On the relativity of uniform rotation. Indian J. Theoret. Phys. 7 (1959), 33–40.

Author's summary: "The meaning of uniform rotation is properly analysed and clarified in the light of the principle of relativity. It is shown that uniform rotation is not an absolute motion and there exists a special relativity of uniform rotation."

3540:

Zatzkis, Henry. The Thomas precession. J. Franklin Inst. 269 (1960), 268–273.

The Thomas precession (i.e., the fact that the resultant of two Lorentz transformations is a single Lorentz transformation followed by a rotation) is derived by a straightforward matrix calculation. *C. W. Kilmister* (London)

3541:

Guggenheimer, H. W. General relativity and nuclear reactions. Dialectica **14** (1960), 183–187.

In a very speculative and not altogether clear manner, the author examines the effect of a highly distorted metric field on the meson cloud surrounding a nucleon.

P. G. Bergmann (Syracuse, N.Y.)

3542:

Newman, Ezra T.; Janis, Allen I. Rigid frames in relativity. Phys. Rev. (2) **116** (1959), 1610–1614.

The authors define as a rigid frame of reference in general relativity a coordinate system such that the spatial distance between two world lines x^k = constant is constant along the world lines (i.e., for all values of x^0 on either curve). They show that this definition is equivalent to N. Rosen's [Phys. Rev. (2) **71** (1947), 54–58; MR **8**, 411]. They point out that such rigid frames do not exist in all Riemannian spaces, but that they can always be constructed in first approximation (i.e., in weak-field approximation).

P. G. Bergmann (Syracuse, N.Y.)

3543:

Misner, Charles W.; Putnam, Peter. Active gravitational mass. Phys. Rev. (2) **116** (1959), 1045–1046.

According to the late R. C. Tolman [*Relativity, thermodynamics and cosmology*, Clarendon, Oxford, 1934], the effective source strength of an electromagnetic radiation field equals twice its energy density; this result would seem to lead to a paradox in that it appears to be in contradiction to the principle of equivalence of gravitational and inertial mass. The authors discuss, and resolve, the paradox by pointing out that the total mass of a confined system must include contributions by the confining walls, which sustain the radiation pressure; they show that this contribution just cancels out the excess gravitating mass of the photon gas.

P. G. Bergmann (Syracuse, N.Y.)

3544:

Bohm, David; Hillion, Pierre; Takabayasi, Takehiko; Vigier, Jean-Pierre. Relativistic rotators and bilocal theory. Progr. Theoret. Phys. **23** (1960), 496–511.

The work deals with a relativistic rotator, defined as a moving point to which is attached a rotating tetrad. Such a model has been proposed by C. Møller [Ann. Inst. H. Poincaré **11** (1949), 251–278; MR **12**, 292] and developed by D. Bohm and J.-P. Vigier [Phys. Rev. (2) **109** (1958), 1882–1891; MR **20** #3018] to provide an appropriate description of an extended particle. The behavior of such a rotator is discussed with the help of a general Lagrangian, and it is shown that there exists a second point which moves in a particularly simple manner, so that the model is similar to that of the bilocal theory of H. Yukawa [ibid. (2) **77** (1950), 219–226; MR **11**, 567]. A special form of Lagrangian is investigated. The motion obtained is regarded as providing a classical analog of the quantum-theoretical relation between energy and frequency. Relativistic Euler angles are introduced as internal variables, these being convenient in the quantization of the theory.

N. Rosen (Haifa)

3545:

Sen, P. Field metrics. Nuovo Cimento (10) **15** (1960), 513–518. (Italian summary)

Author's summary: "A correlation between the metric along which a field propagates and its commutation relations or Feynman propagation function is postulated and the metrics for the fields whose wave equations are known or the wave equations for the fields whose metrics are known are deduced. The metrics of known fields are seen to have simple quadratic forms which define the interaction terms uniquely and from such metrics the interaction terms for nucleon meson, β decay and μ decay interactions are derived."

G. L. Clark (London)

3546:

Plebański, Jerzy; Bażański, Stanisław. The general Fokker action principle and its application in general relativity theory. Acta Phys. Polon. **18** (1959), 307–345.

The authors treat the combination of the gravitational field of general relativity with a set of dynamical variables that represent matter, taking as their point of departure an action principle whose Lagrangian density depends both on the metric tensor and on the dynamical variables describing the matter. Varying with respect to all of these variables, they obtain both the field equations of general relativity and the equations determining the dynamic behavior of matter. The authors point out that the field equations of the gravitational field cannot be solved for arbitrary assumed behavior of the sources, because of the contracted Bianchi identities, which impose on the sources of the gravitational field the condition that the covariant divergence of the matter tensor vanish. Nevertheless, the system of equations can be attacked within the framework of the EIH method (a weak-field, slow-motion approximation), as shown in the paper. The authors apply their general scheme to a special model, in which the matter present is a perfect fluid, or consists of isolated drops of perfect fluid. This paper elucidates some of the basic relationships between the Einstein-Infeld-Hoffmann and the Fock approaches to the problem of ponderomotive laws in general relativity.

P. G. Bergmann (Syracuse, N.Y.)

3547:

Gábos, Z. Contributions à l'étude de l'interaction gravitationnelle des corps matériels. Nuovo Cimento (10) **15** (1960), 395–407. (Italian summary)

After some speculation concerning the Lagrange function of two particles, the author re-derives the well-known expression for this function [which may be obtained with all mathematical rigor starting from first principles—see, for instance, Infeld, Rev. Mod. Phys. **29** (1957), 398–411; MR **19**, 1140; and Plebański and Bażański, preceding review]. The arguments which are used do not pretend to any rigor and are of an heuristic character. Some consideration concerning the Lagrangian for rotating bodies is also given.

J. Plebański (Warsaw)

3548:

Bel, Louis. Champ de gravitation avec induction. C. R. Acad. Sci. Paris **250** (1960), 2137–2139.

The analogy of induction in electromagnetism is

pursued in gravitation by defining a gravitational field $\bar{g}_{\mu\nu}$ related to the geometrical (metric) field by

$$\bar{g}_{\mu\nu} = \Pi_\mu{}^\alpha \Pi_\nu{}^\beta g_{\alpha\beta}.$$

The field equations are

$$F_{\lambda\mu(\alpha\beta|\gamma)} = 0, \quad G_{\lambda\mu}{}^{\alpha\beta}{}_{|\alpha} = U^\beta{}_{\mu|\lambda} - U^\beta{}_{\lambda|\mu},$$

where bars are covariant differentiation in the $\bar{g}_{\mu\nu}$ sense,

$$F_{\lambda\mu\alpha\beta} = \bar{R}_{\lambda\mu\alpha\beta}, \quad G^{\lambda\mu\alpha\beta} = \gamma \bar{R}^{\lambda\mu\alpha\beta}$$

suffixes being raised and lowered by $g_{\mu\nu}$ and

$$U^{\beta\mu} = T^{\beta\mu} - \tfrac{1}{2}(\bar{g}_{\rho\sigma} T^{\rho\sigma})\bar{g}^{\beta\mu}.$$

For example $\Pi_\sigma{}^\rho = \delta_\sigma{}^\rho + (\kappa - 1)u^\rho u_\sigma$ gives a case of fluid flow, $\bar{g}_{\alpha\beta} = g_{\alpha\beta} + (\kappa^2 - 1)u_\alpha u_\beta$. *C. W. Kilmister* (London)

3549:

Hoffmann, Banesh. The Einstein tensor in orthogonal coordinates in n dimensions. J. Math. Mech. **9** (1960), 197–201.

Explicit compact forms of the components of the Einstein tensor, in an orthogonal coordinate system, are derived for an n-dimensional Riemannian space of arbitrary signature. {In equation (27) the second term inside the brackets, $g_{\lambda/\mu} g_{\mu/\mu}$, should be corrected to read $g_{\mu/\lambda} g_{\lambda/\lambda}$.} *C. Gilbert* (Newcastle-upon-Tyne)

3550:

Peres, A. Gravitational radiation. Nuovo Cimento (10) **15** (1960), 351–369. (Italian summary)

Einstein's equations for a system of freely gravitating pole particles are solved by a method of successive approximations. The author emphasizes that in order to calculate the radiation rate one must ensure that at every stage of the approximation there is no external radiation field. This is a difficult problem as the relevant equations are non-linear. A solution is proposed which uniquely specifies the field up to the seventh order (radiation occurs first in the fifth order). The resulting radiation rate is the same as that given by the linearized theory, in the case of two particles revolving on circular orbits (in the Newtonian approximation). (The "anti-damping" previously found by the author [Nuovo Cimento (10) **11** (1959), 644–655; MR **21** #3253] is attributed to the effect of incoming gravitational waves which are here excluded.)

Various properties of the solution are then discussed, such as its behavior at large distances (where it has a static part $\sim m/R$, a quasi-static part $\sim (m/R)^{5/4}$ and a dynamic part $\sim (m/R)^{13/8}$), and in the wave zone.

D. W. Sciama (Ithaca, N.Y.)

3551:

Eisenhart, Luther P. The cosmology problem in general relativity. Ann. of Math. (2) **71** (1960), 384–391.

The usual Friedmann-Robertson-Walker line element is derived from the assumptions of homogeneity and isotropy of space-time, using Killing's equations.

F. A. E. Pirani (London)

3552:

Bonnor, W. B. The problem of evolution in general relativity. J. Math. Mech. **9** (1960), 439–444.

It is proved that a certain space-time, which corresponds to a distribution of perfect fluid, can be fitted continuously onto the flat space-time of special relativity at a preselected instant. The evolution of the perfect fluid need not be uniquely determined in the past, since a transition to flat space-time is thus possible. The perfect fluid, in the example chosen, has a brief period during which its pressure is negative. Apart from this, the space-time closely resembles that of the Einstein-de Sitter model universe. A second example is worked out in which negative pressures are avoided. Thus there are models of the universe whose evolution is indeterminate even if the pressure is always positive and the density-equivalent of the pressure does not exceed one-third of the material density. *G. C. McVittie* (Urbana, Ill.)

3553:

Kurşunoğlu, B. Relativity and quantum theory. Nuovo Cimento (10) **15** (1960), 729–756. (Italian summary)

A formal programme for a unified field theory including quantum physics. The starting point is the author's [Phys. Rev. (2) **88** (1952), 1369–1379; MR **14**, 805] modification of Einstein's final theory, which involves the introduction of a constant of the dimensions of length.

C. W. Kilmister (London)

3554:

Tonnelat, Marie-Antoinette. Généralisation des équations du champ unifié asymétrique. C. R. Acad. Sci. Paris **250** (1960), 2327–2329.

Generalizations of the non-symmetric unified field theory are considered in which the Lagrangian is taken as $L = L_0 + L_1$ with $L_0 = G^{\mu\nu} K_{\mu\nu}$ and $\partial L_1 / \partial g^{\mu\nu} = -\chi(\sqrt{-g}) T_{\mu\nu}$. Here $K_{\mu\nu}$ is a non-symmetric tensor which is a function of the affine connection and its first derivatives, and $T_{\mu\nu}$ is an "energy tensor". *W. B. Bonnor* (London)

3555:

Capella, Alphonse. Sur la quantification du champ unitaire en théorie de Jordan-Thiry à l'approximation linéaire. C. R. Acad. Sci. Paris **250** (1960), 2140–2142.

This paper carries out the quantization of the linearized Jordan-Thiry unitary field theory up to the formulation of Lorentz-covariant commutation relations, in isothermic coordinates. The procedures are conventional, in the formulation used by Lichnerowicz.

P. G. Bergmann (Syracuse, N.Y.)

3556:

Kalicin, Nikola St. On a unified field theory. Izv. Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. **7** (1959), 219–237. (Bulgarian. Russian and English summaries)

From the summary of the author: "The new unified theory about the field submitted here represents a generalisation of Kalutsa's, Klein's and Einstein's 5-dimensional unified field theory for a n-dimensional Riemannian space where n tends to infinity; designate it by R_∞. The 4-dimensional Riemannian space of the general relativity theory is designated by R_4. The fact that the macrophysical phenomena occur in R_4 is expressed in the assumption that R_∞ is cylindrical with respect to the 5th, 6th, etc., dimensions. The element of

the arc and the metric tensor in R_∞ are given by $-ds^2 = g_{ik}dx^i dx^k$. The elements of the arc and the metric tensor in R_4 are given by

$$-d\sigma^2 = \gamma_{\beta\alpha}dx^\alpha dx^\beta,\ \gamma_{\alpha\beta} = g_{\alpha\beta} - g_{\mu\alpha}g_{\mu\alpha} = g_{\alpha\beta} - \Phi_{\mu\alpha}\Phi_{\mu\beta}.$$

The equation of field is obtained from the variational principle

$$\delta(S_f + S_m) = 0,$$

where S_f and S_m are the actions of field and matter. The variational principle leads to the equation

$$\bar{R}^{\alpha\beta} - \tfrac{1}{2}\gamma^{\alpha\beta}\bar{R} - kS^{\alpha\beta} = \alpha c^2 \rho u^\alpha u^\beta,$$

$$(1)\qquad \frac{1}{\sqrt{-\gamma}}\frac{\partial((\sqrt{-\chi})\Phi_\mu{}^{\beta\alpha})}{\partial\chi^\beta} = \alpha c^2 \rho A_\mu u^\alpha,$$

$$(2)\qquad A^\mu = u^\mu + \Phi_{\mu\alpha}u^\alpha = \text{const.}$$

There

$$S_{\alpha\beta} = \frac{2}{k}(\Phi^\delta_{\mu,\alpha}\Phi_{\mu,\delta\beta} + \tfrac{1}{4}\gamma_{\alpha\beta}\Phi^\gamma_{\mu,\delta}\Phi^\delta_{\mu,\gamma}),$$

k is the Einstein gravitational constant. When all $g_{\mu\alpha} = \Phi_{\gamma\alpha}$ and all A^μ are equal to one another equations (1) and (2) pass into the basic equations of relativistic electrodynamics.

"When $g_{\mu\alpha}$ and A^μ differ equations (1) do not coincide with the basic equations of relativistic electrodynamics and can present a system of n photons which is observed in cosmic radiation according to Stein, Caplon and Ritson and which cannot be explained on the basis of the quantum field theory." *M. Pinl* (Cologne)

3557:

Rastall, Peter. An approach to a theory of gravitation. Canad. J. Phys. **38** (1960), 975–982.

In this paper the author constructs a speculative theory of gravitation that gets along on one scalar potential. He assumes the validity of special relativity. Because the author's theory is admittedly incomplete, he has plenty of adjustable parameters to fit the outcome of the well-known three types of observations, i.e., the perihelion precession of planetary orbits, the deflection of light by a gravitating body, and the gravitational red shift. The principle of equivalence is not discussed.

P. G. Bergmann (Syracuse, N.Y.)

3558:

Hoffmann, Banesh. On Yilmaz' new approach to general relativity. J. Math. Mech. **9** (1960), 445–451.

In a new theory of gravitation presented by Yilmaz [Phys. Rev. (2) **111** (1958), 1417–1426; MR **20** #5066], the metric tensor $g_{\mu\nu}$ was regarded as a functional of a scalar function ϕ. Hoffmann now shows that a generalised form of Yilmaz' theorem is valid for a line-element of the form

$$ds^2 = e^{-2(n-3)\phi}(dx^0)^2 + e^\phi g_{ab}dx^a dx^b,$$

where the indices a and b run from 1 to $n-1$ and both ϕ and g_{ab} are independent of x^0, provided the curvature tensor $R_b{}^a$, belonging to the g_{ab}, vanishes. Whereas in the space-time case, the sub-metric is three-dimensional and the condition $R_b{}^a = 0$ is equivalent to the vanishing of the curvature tensor, in the general case the condition does not imply flatness. Since in certain unified field theories spaces of five dimensions are considered it is suggested that the theory may give interesting results in such cases and that it may be necessary to give a different interpretation of the role of the ϕ field.

A number of difficulties connected with Yilmaz' theory are also discussed. In particular, Yilmaz' derivation of the relation $E = M_g c^2$ is shown to be incorrect.

G. L. Clark (London)

ASTRONOMY

See also 3150, 3374, 3551.

3559:

Kopal, Zdeněk. ★Figures of equilibrium of celestial bodies: with emphasis on problems of motion of artificial satellites. Publ. No. 3, Mathematics Research Center, United States Army, Univ. of Wisconsin. The University of Wisconsin Press, Madison, 1960. vi+135 pp. $3.00.

A presentation of the first-order solution for the equilibrium of a self-gravitating rotating body and its tidal deformation serve as an introduction. The principal object of the volume is the development of a theory for the form and potential of a fluid body carried to quantities of the second order in tesseral harmonics arising from rotation or tides. In Chapter III the second-order theory is given separately for rotation and tidal deformation. In Chapter IV the more difficult interaction phenomena are included. These chapters contain a more complete and orderly treatment of this problem than had been given by previous authors. A chapter on non-radial oscillations contains a brief treatment of both free and forced oscillations that must be considered in non-equilibrium problems. *D. Brouwer* (New Haven, Conn.)

3560:

★Séminaire de mécanique analytique et de mécanique céleste dirigé par Maurice Janet. 2e année: 1958/59. Faculté des Sciences de Paris. Secrétariat mathématique, 11 rue Pierre Curie, Paris 5e, 1959. 173 pp. (mimeographed)

Le volume constitue une série de conférences du Séminaire de mécanique analytique et mécanique céleste dirigé par M. Janet à la Faculté des Sciences de Paris, année 1958–1959.

La première conférence de Y. Thiry montre qu'on peut représenter le champs électromagnétique et le champs de gravitation par un seul tenseur, le tenseur métrique d'une variété riemannienne à cinq dimensions, la cinquième dimension n'ayant aucune signification physique. Dans la seconde conférence, Mlle J. Renaudie prolonge les recherches de Thiry en introduisant dans le schéma riemannien une dimension en plus et définit par le tenseur métrique V_6 un champ d'une particule de spin maximum 1 couplé avec le champs de gravitation. Dans la troisième conférence, J. Gremillard expose la méthode de H. Poincaré pour la recherche de solutions périodiques dans le problème des 3 corps, et il démontre que sous certaines conditions, il existe des solutions de 3ème sorte (à inclinaisons non nulles quoique petites). Dans la quatrième et cinquième conférences O. Costa de Beauregard étudie l'effet

inertial du spin et l'hypothèse de l'effet gravitationnel du spin. Dans la sixième conférence de E. Schatzman sur les problèmes cosmogoniques (confrontation de la théorie avec les observations), l'auteur étudie le décalage vers le rouge des raies spectrales et l'âge de l'univers. Dans la septième conférence de M. Henon, l'auteur étudie la dynamique des amas d'étoiles. Dans la huitième, C. B. Rayner étudie le problème de corps rigides en relativité générale. Dans la 9ème conférence J. Colleau étudie en détail un contre exemple de Sitnikov qui paraît contredire certains résultats obtenus par J. Chazy dans l'allure extrême du mouvement des 3 corps. Dans la 10ème conférence, J. Meffroy présente une étude sur l'expression analytique et le calcul effectif du terme séculaire pur de la perturbation du 3ème ordre des grands axes. Dans la 11ème conférence, A. Lichnerowicz reprend la théorie des ondes et radiations électromagnétiques de la relativité restreinte et montre comment cette théorie peut être adaptée à la relativité générale. Cette adaptation lui sert dans une seconde partie de son travail pour l'étude du cas gravitationnel. Dans la 12ème conférence, la radiation gravitationnelle, L. Bel montre en suivant le point de vue de Lichnerowicz et Pirani que c'est le tenseur de courbure plutôt que le tenseur métrique qui constitue l'élément géométrique adéquat à la description des phénomènes de radiation gravitationnelle. *M. Kiveliovitch* (Paris)

3561:

Gall, Ruth; Lifshitz, Jaime. A new computation of simple shadow cones. Rev. Mexicana Fis. **9** (1960), 57–68. (Spanish. English summary)

Authors' summary: "Simple shadow cones for ten points of incidence have been evaluated for several latitudes, longitudes and proton energies. The trajectories were computed in the earth's dipole and quadrupole magnetic field. The purpose of this new computation was (a) to calculate the correct simple shadow cones and estimate the error in the cones calculated by Schremp in 1938 and (b) to study the longitude effect of the simple shadow cones.

"A new computation of simple shadow cones has also recently been made by J. E. Kasper from the University of Iowa. The cones calculated by Kasper and Schremp present no longitude effect, as these authors considered the motion of charged particles in the magnetic field of a central dipole.

"The agreement between Kasper's and our cones and the discrepancy with the Schremp cones demonstrates that the Schremp cones are erroneous and predict a shadow effect of the earth larger than the correct one. The discrepancy between our cones and Schremp's is mostly due to the error in the Schremp calculations and only in a smaller degree to the longitude effect.

"The comparison between Kasper's and our cones allows one to estimate the longitude effect. This effect is important at low latitudes for small particle energies and decreases with the increase of latitude. The effect of the earth's shadow is smaller for the considered smaller longitude. The shadow cone of the dipole field falls between the two cones corresponding to two different longitudes of the dipole and quadrupole magnetic field.

"The limitations of the method of integration when used for higher energies are discussed."

3562:

Fialko, E. I. A method for studying the distribution by mass of meteoric bodies. Astr. Ž. **36** (1959), 1058–1060 (Russian. English summary); translated as Soviet Astr. AJ **3** (1960), 970–973.

Author's summary: "The method outlined may be used to study the masswise distribution pattern of meteoric bodies, by utilizing the distribution by duration of persistent radio echoes returned from meteor trails. Experimental results from the application of this method to the Perseid shower are presented."

3563:

Idlis, G. M. Relation between the general properties of the gravitational potential of systems of stars and the general nature of the integrals of motion of a particular star. Akad. Nauk Kazah. SSR. Izv. Astr. Inst. **8** (1959), 24–52. (Russian. English summary)

Author's summary: "The analysis of a possible quantity and of a general form of simple motion integrals of an individual star which can go as independent arguments in a phase density of a stellar system, shows that in the general case the number of these integrals is three, of which the first is an energy integral which is bound with stability of a system, the second is a moment integral, guarantees a rotation symmetry of a system and permits its differential axial rotation and the third has sufficiently arbitrary form, but ensures a three-axial distribution of stellar peculiar velocities and a symmetry of a system relative to an equatorial plane, where a gravitational potential comes to a generalized Parenagós formula in the first approximation."

3564:

Henon, M. Un calcul amélioré des perturbations des vitesses stellaires. (Problème du "temps de relaxation".) Ann. Astrophys. **21** (1958), 186–216. (English and Russian summaries)

Une étoile-test de masse M, de vitesse $\mathbf{V}$, pénètre dans un champ d'étoiles dont le mouvement est décrit par les procédés statistiques. Au bout du temps T, sa vitesse s'accroît de $\Delta\mathbf{V}$. L'auteur considère ce vecteur comme une quantité aléatoire et calcule ses moments des 2 premiers ordres. Il utilise la méthode d'approximation exposée par Chandrasekhar dans son traité *Principles of stellar dynamics* [University of Chicago, Chicago, Ill. (1942); MR **4**, 57]. Soit $\mathbf{E}$ la perturbation subie par l'étoile-test en présence d'une étoile de champ unique; on pose: $\Delta\mathbf{V} = \sum \mathbf{E}$, la sommation étant étendue à toutes les étoiles de champ.

Chandrasekhar calculait la perturbation élémentaire $\mathbf{E}$ en supposant que chacune des 2 étoiles en "rencontre" décrivait une orbite hyperbolique complète autour du centre de gravité commun. Il surestimait ainsi l'influence des étoiles lointaines et trouvait de ce fait des intégrales divergentes, qu'il coupait arbitrairement à une certaine distance D_0.

L'auteur reprend ce calcul. Il adopte la méthode de Chandrasekhar pour les étoiles proches. Pour les étoiles lointaines, il pose $\mathbf{E} = M^{-1} \int_0^T \mathbf{F} dt$ et calcule la force d'attraction $\mathbf{F}$ des 2 étoiles en "rencontre" comme si celles-ci décrivaient les trajectoires linéaires non perturbées. Il trouve ainsi des intégrales convergentes,

semblables analytiquement aux expressions de Chandrasekhar. Il signale une application aux problèmes analogues des plasmas totalement ionisés.

Le reviewer présente les critiques suivantes. Pour les étoiles proches, une rencontre est improbable; mais lorsqu'elle se produit elle bouleverse complètement le mouvement de l'étoile-test; l'approximation $\Delta \mathbf{V} = \sum \mathbf{E}$ ne semble pas correcte. Pour les étoiles lointaines, le calcul de la perturbation revient à intégrer l'expression $\int_0^T \mathbf{F}\, dt$ par rapport à toutes les étoiles de champ. Un changement dans l'ordre des intégrations conduit à l'approximation classique de la dynamique stellaire, qui consiste à remplacer l'attraction des étoiles lointaines par l'attraction d'un milieu continu. *F. Nahon* (Marseilles)

3565:

Idlis, G. M. The symmetry of stationary axially symmetric stellar systems relative to the equatorial plane. Soviet Astr. AJ **3**, 88–91 (1959).

English translation of Astr. Ž. **36** (1959), 85–88 [MR **21** #1204.]

3566:

von Hoerner, Sebastian. Die zeitliche Rate der Sternentstehung. Fortschr. Physik 8 (1960), 191–244.

3567:

Gábos, Zoltán. Contributions à l'étude de l'interaction gravitationnelle des corps sphériques en rotation. An. Şti. Univ. "Al. I. Cuza" Iaşi. Şect. I (N.S.) **5** (1959), 101–112. (Romanian. Russian and French summaries)

Author's summary: "L'auteur démontre que la fonction de Lagrange, concernant l'interaction gravitationnelle des corps en rotation, peut être donnée par une méthode directe (sans que le théorème du moment de la quantité de mouvement soit utilisé), en utilisant l'expression de la fonction de Lagrange, L, établie par I. G. Fichtengoltz pour le cas du problème des n-points matériels."

3568:

Grémillard, Jean. Recherches sur les conditions d'existence de solutions périodiques de la troisième sorte du problème des trois corps. Bull. Astr. **22** (1959), 221–332. (English, German and Russian summaries)

Besides the von Zeipel memoir [Nova Acta Soc. Sci. Upsal. Sect. I (3) **20** (1904), no. 9] this is the most extensive treatment concerning the existence of the Poincaré periodic solutions of the third kind in the three-body problem. A condition for their existence is the commensurability of the mean motions n and n' of the two bodies moving around the central body, i.e. the relation $n/n' = p/q$, where p and q are relatively prime positive integers, is to be satisfied. If the mutual inclination of the osculating orbits is small, the method used by the author holds, whatever is the parity of $p-q$. If, however, the mutual inclination of the orbits takes an arbitrary value between 0° and 180°, a development of the perturbation function different from that used in the first case is required. As a consequence of this fact the discussions to which the author is led differ greatly as to whether $p-q$ is an even or odd integer. It is shown that periodic solutions of the third kind exist only for some particular initial values of the osculating elements of the two orbits or some related variables. The author has also elucidated and examined more thoroughly certain not explicitly elaborated statements of the above memoir by von Zeipel [cf. also Grémillard, C. R. Acad. Sci. Paris **234** (1952), 2339–2341; **236** (1953), 49–51, 1952–1953; **239** (1954), 153–155; MR **13**, 996; **14**, 802, 1132; **16**, 181].

E. Leimanis (Vancouver, B.C.)

3569:

★Proceedings of Symposia in Applied Mathematics. Vol. IX: Orbit theory. American Mathematical Society, Providence, R.I., 1959. v+195 pp. $7.20.

The 10 papers in this volume will be reviewed separately.

3570:

Stovas, M. V. Deformation of ellipsoid parameters as a result of variation of ellipticity. Vestnik Leningrad. Univ. **14** (1959), no. 13, 121–136. (Russian. English summary)

3571:

Zhevakin, S. A. On the calculation of nonadiabatic stellar pulsations by use of a discrete model. Soviet Astr. AJ **3**, 267–279 (1959).

English translation of Astr. Ž. **36** (1959), 269–282 [MR **21** #1896].

3572:

Porfir'ev, V. V. The law of rotation of a polytropic gas sphere. Soviet Astr. AJ **3**, 531–532 (1959).

English translation of Astr. Ž. **36** (1959), 546–549 [MR **21** #4047].

3573:

Minin, I. N. Non-stationary problems of radiation transfer theory. Vestnik Leningrad. Univ. **14** (1959), no. 13, 137–141. (Russian. English summary)

3574:

Kreiken, E. A. Outline of a method for determining the absorption in the surroundings of the sun. Comm. Fac. Sci. Univ. Ankara Sér. A **10** (1959), 1–25. (Turkish summary)

Author's summary: "In the following a method is proposed to determine the mean coefficient of interstellar absorption and the mean extinction from the observed spectroscopic parallaxes, the radial velocity and the proper motions. The distribution $\Phi(\dot{x}, \dot{y}, \dot{z}/S)$ is assumed to be constant in the immediate surroundings of the sun. This is a reasonable assumption. Therefore the velocity law must be a constant along any axis, independent of its orientation. In different areas of the sky it is convenient to adopt as axis the radius vector and the direction perpendicular to and tangent to the circles α and δ. The equations of motion containing a are derived. From these the numerical values of a can easily be derived."

3575:

Sobolev, V. V. Selected problems in the theory of radiation diffusion. Dokl. Akad. Nauk SSSR **129** (1959), 1265–1268 (Russian); translated as Soviet Physics. Dokl. **4** (1960), 1235–1238.

3576:

Stibbs, D. W. N.; Weir, R. E. On the *H*-functions for isotropic scattering. Monthly Not. Roy. Astr. Soc. **119** (1959), 512–525.

La fonction $H(\mu)$ définie par l'équation intégrale non linéaire (1) est extrèmement importante pour la théorie du transfert radiatif.

La fonction $H(\mu)$ définie par

$$(1) \qquad H(\mu) = 1+\mu H(\mu) \int_0^1 \frac{\psi(\mu')}{\mu+\mu'} H(\mu')d\mu'$$

peut être mise dans le cas $\psi(\mu)=\text{constante}=1/2\varpi$ sous la forme $H(\mu, \varpi)=\exp[I(\mu, \varpi)]$ avec

$$I(u, \varpi) = -\frac{\mu}{\pi}\int_0^{\pi/2} \frac{\log_e(1-\varpi\theta\cot\theta)}{\cos^2\theta+\mu^2\sin^2\theta}\,d\theta.$$

Stibbs et Weir donnent la fonction $H(\mu, \varpi)$ pour $\mu=0$ (0,05) 1.0 pour 30 valeurs de ϖ sélectionnées parmi 160 valeurs $\varpi=0$ (0.01) 0.90 (0.005) 0.95 (0.001) 0.99 (0.0005) 1.0.

Des représentations polynomiales de $H(\varpi)$ sont données.

E. Schatzman (Paris)

3577:

Öpik, E. J.; Singer, S. F. Distribution of density in a planetary exosphere. Phys. Fluids **2** (1959), 653–655.

Author's summary: "A theory has been developed which gives the distribution of density with altitude for a planetary exosphere in the absence of local thermodynamic equilibrium. It gives values considerably lower than those conventionally calculated on the basis of the hydrostatic equation. Our results apply to the case where the field of force is gravitational; hence in the case of the earth, they give the density variation of only the neutral component of the exosphere."

3578:

Grzędzielski, S. On gas expansion in the central parts of the galaxy. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 627–632. (Russian summary, unbound insert)

Étude du mouvement des gaz neutres et ionisés dans un système de révolution dans les hypothèses suivantes: (1) le gaz ionisé est entraîné par le champ magnétique; (2) le mouvement du gaz neutre est dû à la friction du gaz ionisé; (3) cette friction est celle de sphères immergées dans un fluide compressible et se déplaçant à une vitesse supersonique. Le freinage du mouvement radial du gaz neutre est freiné par cette interaction et produit une diminution brusque de la vitesse d'expansion vers 4 kpc.

L'importance de la friction des atomes d'hydrogène sur les électrons des régions H I (un électron pour 10^4 atomes neutres) n'a pas été examinée. *E. Schatzman* (Paris)

3579:

Jensen, Eberhart. On the dynamics of prominences and coronal condensations. Astrophys. Norvegica **6**, 93–111 (1959).

Pour un gaz sans collisions plongé dans un champ magnétique ayant la symétrie cylindrique et dont l'échelle de variation est grande comparée au rayon de Larmor, E. Jensen écrit l'équation de Boltzmann, en supposant que la fonction de distribution des vitesses a la symétrie cylindrique. Suivant que la différence $E_\perp-2E_\parallel$ entre l'énergie cinétique dans les directions perpendiculaire et parallèle au champ magnétique est positive ou négative le plasma est diamagnétique ou paramagnétique. Bien que les effets d'induction n'aient pas été étudiés, l'auteur pense pouvoir expliquer certains mouvements coronaux par des variations rapides du champ magnétique, d'échelle de temps inférieure au temps de relaxation des anisotropies. Ce temps de relaxation des anisotropies est pris égal à l'intervalle de temps entre deux collisions.

E. Schatzman (Paris)

3580:

Agekyan, T. A. The velocity distribution function and the rate of dissipation in systems of gravitating bodies. Soviet Astr. AJ **3**, 280–290 (1959).

English translation of Astr. Ž. **36** (1959), 283–294 [MR **21** #2524].

3581:

Agekyan, T. A. The probability of a stellar approach with a given change of the absolute velocity. Soviet Astr. AJ **3**, 46–58 (1959).

Translation of article in Astr. Ž. **36** (1959), 41–53 [MR **21** #7791].

3582:

Nariai, Hidekazu; Ueno, Yoshio. On a new approach to cosmology. II. Progr. Theoret. Phys. **23** (1960), 305–327.

A sequel to a previous paper [Prog. Theoret. Phys. **21** (1959), 219–231; MR **21** #1213] extending the discussion to include local gravitation.

D. W. Sciama (Ithaca, N.Y.)

3583:

Davidson, W. Number count relations in observational cosmology. Monthly Not. Roy. Astr. Soc. **119** (1959), 665–681.

From the author's summary: "Theoretical formulae relating the counts of sources of radiation to other observable quantities in cosmology are presented in the form considered most relevant to present-day problems, the analysis being based on the Robertson-Walker space-time metric for a 'smoothed out' universe. The relations are designed to apply to the number counts of sources emitting in either the optical range of the spectrum or in the radio range, and these are determined in terms of both red shift and apparent magnitude. An alternative number-count formula is derived which may possibly be more applicable in an evolutionary universe should radio sources arise by the collision of galaxies. Each relation is discussed in some detail with regard to the value of its application to the actual universe. The paper concludes

with a systematic comparison between the observable number-count characteristics of an evolutionary universe and those of the steady-state model, indicating several means of distinguishing between these two types of universe." *A. G. Walker* (Liverpool)

3584:

Ahmavaara, Yrjö. The group theoretical secture of the physical world. Arkhimedes **1959**, no. 1, 21–30. (Finnish)

3585:

Johler, J. Ralph; Walters, Lillie C. On the theory of reflection of low- and very-low-radiofrequency waves from the ionosphere. J. Res. Nat. Bur. Standards Sect. D **64D** (1960), 269–285.

The equations of the magneto-ionic theory are applied to the case of a plane wave

$$|\bar{E}_i| \exp\{i[\omega t - \frac{\omega}{c}(x \sin\phi_i \sin\phi_a + y \sin\phi_i \cos\phi_a + z\cos\phi_i)]\}$$

incident on the plane boundary of a uniform model ionospheric layer. The transmitted wave is assumed to have the form

$$|\bar{E}_t| \exp\{i[\omega t - \frac{\omega}{c}(x \sin\phi_i \sin\phi_a + y \sin\phi_i \cos\phi_a + z\zeta)]\}.$$

Then the conditions of propagation are determined by the values of ζ, which satisfies a quartic equation. The four roots represent upgoing and downgoing, ordinary and extraordinary waves. The upgoing pair and the boundary conditions determine the reflection and transmission coefficients for the incident wave.

General results of this procedure suitable for electronic calculation are given, and from them the quasi-longitudinal approximations are derived, using Snell's law. A series of curves representing reflection coefficients for model D and E regions are given together with some of those derived from the quasi-longitudinal approximation. *K. C. Westfold* (Sydney)

3586:

Singh, R. N.; Murty, Y. S. N. Dispersion, absorption and polarization curves for radio wave propagation through the ionosphere in the presence of earth's magnetic field. J. Sci. Res. Banaras Hindu Univ. **9**, no. 1, 1–18 (1958/59).

Authors' summary: "The absorption, dispersion and polarization curves are drawn under the magnetic conditions of ionosphere above Banaras (latitude 25° 18′ 25″ N, longitude 83° 0′ 46″ E, Dip angle 36° 26′ N, and $H=0.446$ Gauss) for the normal-incidence of radiowave propagation. These curves are obtained by applying Bailey's method of conformal representation to the Appleton-Hartree formulae and refer to four typical wave-frequencies (30m, 100m, 240.1m and 400m) and to four values of electron collisional frequency ($\nu=0$, $\nu=10^6$, $\nu=\nu_c=4.49\times10^6$ and $\nu=10^7$) which are likely to cover the range of practical importance for the radiowave propagation. Curves for both the magnetoionic components namely ordinary and extraordinary were drawn considering the variation of the earth's magnetic field with height of the E-layer and F-layer. The ψ-curves for polarization were drawn by using a new convention which states that the major axes of the ordinary and extraordinary ellipses are not in general mutually perpendicular, but instead have the property that the sum of the tilt-angles which they make with the Y-axis is an odd integral multiple of $\pi/2$. The absorption curves for Banaras show that the ordinary is more absorbed than extraordinary for larger electron collisional frequency." *K. C. Westfold* (Sydney)

GEOPHYSICS

See also 3246, 3301.

3587:

Wassef, A. M. Note on the application of mathematical statistics to the analysis of levelling errors. Bull. Géodésique (N.S.) No. 52 (1959), 19–26.

3588:

Wassef, A. M.; Messih, F. Z. A. On the statistical distribution of levelling errors. Bull. Géodésique (N.S.) No. 56 (1960), 201–210.

The authors attempt to explain to geodesists that a mixture of normal distributions may not be a normal distribution. The presentation of examples of non-normality resulting from the mixture of normal distributions of either unequal variances or unequal means is given in terms of the two questionable classical measures of normality; namely, skewness and kurtosis. *M. Muller* (New York)

OPERATIONS RESEARCH, ECONOMETRICS, GAMES

See also 3098.

3589:

Fels, E. M. Econometrica Sovetica. I, II. Allg. Statist. Arch. **43** (1959), 135–148; **44** (1960), 15–26. (German)

Survey of current activity in econometrics in the U.S.S.R.

3590:

Rajaoja, Vieno. A study in the theory of demand functions and price indexes. Soc. Sci. Fenn. Comment. Phys.-Math. **21** (1958), no. 1, 96 pp.

This book contains firstly an exposition of the theory of indifference maps and demand functions.

Three different types of price indexes are considered, namely, the cost of living index, the marginal price index, and the competitors' index which is defined as follows. Let $p_1^0, \cdots, p_k^0, \cdots, p_n^0, \mu^0$ be prices and income in a reference situation and assume that the prices of the goods $G_1, \cdots, G_{k-1}, G_{k+1}, \cdots, G_n$ change to the values (A) $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_n$. Let the price of G_k and the consumer's income take on the values (B) y_k; μ such that in the price-income situation determined by (A) and

(B), the consumer's real income and the quantity of G_k consumed are the same as in the reference situation. Then the competitors' index of G_k is equal to y_k/p_k^0. It is shown that a demand function $q_i = q_i(p_1, \cdots, p_n, \mu)$ defined from indifference maps can be given the form (C) $q_i = f_i(p_i/P, Q)$, where P is the competitors' index and Q real income. The indexes and differences between them are investigated both theoretically and by numerical examples.

The properties of separable groups of goods are specially examined. The good G_n is called a strong good if the preference function may be written in the form $U = U\{\varphi(G_1, G_2, \cdots, G_{n-1}), G_n\}$. The demand function for a strong good may be written in the form (C) where P is the marginal price index.

The last chapter contains some results concerning demand functions when the consumer is subsidized and in case of rationing with a black market.

S. Malmquist (Stockholm)

3591:

Winters, Peter R. Forecasting sales by exponentially weighted moving averages. Management. Sci. **6** (1960), 324–342.

This paper develops an earlier method of C. C. Holt's for estimating future sales through a weighting of forecasts and actual data. In a similar way the seasonal and trend factors can be used and adjusted. The method has the advantage of simplicity and of requiring no external data (and with it the disadvantage of ignoring other relevant economic information), and that forecasts can be computed quickly.—Three real life examples are given in which the forecasting of errors compares favourably with those found in more detailed economic studies.

G. Morton (London)

3592:

Malcolm, D. G. Bibliography on the use of simulation in management analysis. Operations Res. **8** (1960), 169–177.

3593:

Rose, Alan. A note on the use of logical computers to determine the most efficient method of using factory machines. Proc. Cambridge Philos. Soc. **56** (1960), 186–188.

The present paper is an extension of the author's previous note [same Proc. **54** (1958), 307–321; MR **20** #2285], and the present note cannot be read without the earlier one. The logical mechanism of the earlier note is used here (with mild variations) to deal with a problem involving using various machines to do various jobs in various times. The construction is such that delays and costs of delays (in completions of jobs) are proportional to (m-valued) truth-values of certain formulae.

R. M. Baer (Berkeley, Calif.)

3594:

Levinson, Louis. A theory of mortality classes. Soc. Actuar. Trans. **11** (1959), no. 3, 46–87.

Der Verf. teilt einen Versicherungsbestand, unabhängig vom Alter, entsprechend der Veranlagung in Mortalitätsklassen ein. Nach einem Jahr soll eine gewisse Wahrscheinlichkeit bestehen, dass ein Versicherter in eine Klasse von höherer Sterbenswahrscheinlichkeit eingeteilt wird. Der Verf. berechnet die Kommutationszahlen und dementsprechend auf Grund von Annahmen über die Art der Verteilung seine Theorie an Hand der U.S.-Life Tables 1949/51. Ein grosses Zahlenmaterial wurde zu diesem Zwecke verarbeitet.

Der Ref. hält es für möglich, dass auf Grund dieser interessanten Untersuchung die Sterblichkeitsmessung zukünftig auch nach den Gesichtspunkten erfolgen werde, wie sie vom Verf. vorgeschlagen wurden.

W. Saxer (Zürich)

3595:

Hadley, George F.; Simonnard, Michael A. A simplified two-phase technique for the simplex method. Naval Res. Logist. Quart. **6** (1959), 221–226.

The first phase of the two-phase solution of a linear programming problem consists in obtaining a basic solution $x \geqq 0$ of the linear equations $Ax = b$ $(b \geqq 0)$ by applying the simplex method to the minimization, to zero, of $\sum y_i$ under the constraints $Ax + y = b$, $x \geqq 0$, $y \geqq 0$; this problem has the obvious basic solution $x = 0$, $y = b$. It is proposed that the following technique will ensure the maintenance of $Ax = b$ during the second phase of the solution without requiring that the constraint $\sum y_i = 0$ be added to the system: the "usual" rule for selecting a pivot column and subsequently a pivot row is employed, except that if the pivot column contains no positive, and some negative, entries corresponding to artificial variables (y_i), then the pivot row is chosen as any bearing a negative entry in the pivot column. A numerical example is successfully worked out.

P. Wolfe (Santa Monica, Calif.)

3596:

Dennis, Jack B. A high-speed computer technique for the transportation problem. J. Assoc. Comput. Mach. **5** (1958), 132–153.

A digital computer technique for solving Hitchcock's transportation problem is presented. The stepping-stone method, which forms the basis for this technique, is discussed in some detail. In a problem with m origins and n destinations, many of the mn routes are obviously impractical, so that their specific shipping costs may be regarded as infinite. The specific shipping costs of the remaining routes will only be needed sequentially, so that they may be recorded on magnetic tape. In any basic feasible solution the flows along all but at most $m+n-1$ routes will vanish. Such a solution can therefore be stored in the core memory as a basic table of $m+n-1$ entries giving the row and column numbers and the value of each element of the basis as well as information that facilitates the tracing of basic paths. The major routines (form initial basic feasible solution; search for next element to enter basis; find basic loop including new element; change values of loop elements; modify basis table; compute dual variables for new basis) are illustrated by flow diagrams. To indicate the use of the basis table, the third of these routines is treated in detail. Two methods of selecting the next element to enter the basis are discussed and limited practical experience with these methods is presented.

W. Prager (Providence, R.I.)

3597:

Glicksman, Stephen; Johnson, Lyle; Eselson, Leonard. Coding the transportation problem. Naval Res. Logist. Quart. **7** (1960), 169–183.

To facilitate coding of the Hitchcock transportation problem for a digital computer, the elements of a basic feasible solution are marked as follows. In the row containing a candidate for the next element to be introduced in the basis, all elements of the given basis are marked by circles; in the columns containing these circles all other elements of the basis are marked by squares; in the rows containing these squares, all as yet unmarked elements of the basis are marked by circles, and so on. When the marking is completed, each column will contain one circle and each row a square with the exception of the row of the candidate element (exceptional row). Simple rules are formulated for tracing the loop through the candidate element and adjusting the basis, and for making another row exceptional. In a transportation problem with m origins and n destinations, the basis contains at most $m-1$ squares falling into at most $m-1$ columns. To be involved in a loop, a column must contain either the candidate element or a square. Thus any improvement of a basic feasible solution to an m by n transportation problem can be accomplished in a "nucleus" of m columns. When the number of destinations far exceeds the number of origins, it is expedient to store only the nucleus in the core memory and change it by one column at a time in the course of the computation. A numerical representation of the marking scheme is given, and its use in the adjustment of flows and shadow prices is illustrated by a numerical example. Experience with a Univac 1 code is described.

W. Prager (Providence, R.I.)

3598:

Manne, Alan S. On the job-shop scheduling problem. Operations Res. **8** (1960), 219–223.

Noninterference restrictions occur frequently in job-shop scheduling. For example, jobs j and k cannot occupy the same machine at the same time. Either one of the two jobs must precede the other one. With the classical form of linear programming it is not possible to specify such either-or conditions. The paper states the typical job-shop scheduling problem as a discrete linear programming problem. Noninterference restrictions of the type shown above can be handled but the computational feasibility of this approach for large scale realistic problems has not yet been established.

W. W. Leutert (New York)

3599:

Dantzig, George B. On the significance of solving linear programming problems with some integer variables. Econometrica **28** (1960), 30–44.

The author shows how combinatorial problems can be formulated as pure integer programming problems, and how other problems involving the minimization of non-convex functions over non-convex regions can be formulated as mixed integer programming problems, i.e., problems where some but not all of the variables must take integer values.

Several examples are given.

E. M. L. Beale (Teddington)

3600:

Dufour, H.-M. Résolution du problème fondamental de la recherche opérationnelle par approximations quadratiques. Chiffres **2** (1959), 181–195. (English, German and Russian summaries)

Linear programming as well as non-linear programming problems, stated in terms of inequality (or conditional) relationships, are treated by a novel method. The inequality relationships are replaced by suitable "elastic" relationships (retaining the boundaries of the limitations and constraints imposed on the original variables).

This substitution leads to the formulation of new relationships and a new problem whose solution is equivalent (methodologically) in nature to the classical method of least squares.

The principles enunciated in this paper have been suggested by L. V. Kantorovič in his two works (in Russian): (a) *Mathematical methods of the organization and the planning of production*, Leningrad University Publications, 1939. (b) (In collaboration with M. K. Gavurin) *The mathematical methods of the analysis of the flow of freight*, The Academy of Sciences of USSR, *Problems of the raising of the effectiveness of the work of transportation*, 1953.

A. Bakst (Flushing, N.Y.)

3601:

Rosen, J. B. The gradient projection method for nonlinear programming. I. Linear constraints. J. Soc. Indust. Appl. Math. **8** (1960), 181–217.

The nature of the method is sufficiently well indicated by the name. The method is described in detail, first with nonlinearity only in the objective function, then with nonlinearity allowed also in the constraints. Much of the paper is taken up with an exposition of the rather elementary matrix theory utilized by the method so that only a minimal background is required for an understanding.

A. S. Householder (Oak Ridge, Tenn.)

3602:

Dorn, W. S. Duality in quadratic programming. Quart. Appl. Math. **18** (1960/61), 155–162.

Using the duality theorem of linear programming, the author proves a duality theorem relating the minimization of a convex quadratic function of variables subject to linear (inequality) constraints to the maximization of a concave quadratic function of variables subject to linear constraints.

The author shows that his result is a natural generalization of a classical result for equality-constrained variables; and also applies it to a problem in elasticity, where it represents the principle of virtual work.

E. M. L. Beale (Teddington)

3603:

Karush, William. A theorem in convex programming. Naval Res. Logist. Quart. **6** (1959), 245–260.

Let K be the convex set of all $(x_1, \cdots, x_n)$ satisfying $A \leq x_1 \leq \cdots \leq x_n \leq B$. Let f_i be convex functions of a real variable, and $F(A, B) = \min_K \sum f_i(x_i)$. Then (i) $F(A, B) = M(A) + N(B)$, where M and N are increasing and decreasing convex functions, respectively, and (ii) $F(A, C) = F(A, B) + F(B, C) - F(B, B)$. The author also indicates a proof of the continuous analogues of these results. Finally

he describes an algorithm, based on dynamic programming, for finding $(\bar{x}_1, \cdots, \bar{x}_n) \in K$ such that $F(A, B) = \sum f_i(\bar{x}_i)$. *A. J. Hoffman* (New York)

3604:

Karlin, Samuel. Dynamic inventory policy with varying stochastic demands. Management. Sci. **6** (1960), 231–258.

A dynamic inventory model of the Arrow-Harris-Marschak type [see Econometrica **19** (1951), 250–272; MR **13**, 368] is formulated in which demand is independently but not identically distributed in different periods. The optimal policy at each stage is characterized by a single critical number which also could vary in successive periods. The dependence of the critical numbers as a function of stochastic ordering amongst distributions is developed under various conditions. The probability density $f(x)$, $x \geq 0$, is defined to be stochastically smaller than the density $g(x)$ if, for all $x \geq 0$, $\int_0^x f(t)dt \leq \int_0^x g(t)dt$. The analysis is based on the assumption of linear purchasing cost, with occasional extensions to convex purchasing cost. *M. J. Beckmann* (Providence, R.I.)

3605:

Shapley, L. S. The solutions of a symmetric market game. Contributions to the theory of games, Vol. IV, pp. 145–162. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

Denote by $|X|$ the number of elements in the set X. A "symmetric market game" has $|M|+|N|$ players, "buyers" and "sellers", constituting the sets M and N, and is defined by the characteristic function $v(S) = \min(|S \cap M|, |S \cap N|)$ for $S \subseteq M \cup N$. The main feature of such a game, besides the symmetry of the roles of the members of one of the sets M, N, is the fact that members of a given set cannot enter into profitable coalitions.

Imputations for this game are nonnegative $m+n$-component vectors $x = (x'; x'')$, where x' represents payoffs to the m members of M and x'' payoffs to the n members of N, and whose components sum to $g = \min(m, n)$. A "face" A_S of the $m+n-1$ dimensional simplex of all imputations is the set of all imputations whose components corresponding to members of S sum to one. The principal new notion used in obtaining the results below is that of "skewness": two imputations x, y are "skew" if either $x' \geq y'$ and $x'' \leq y''$, or $x' \leq y'$ and $x'' \geq y''$. A "skew set" is one in which every pair of imputations is skew. It is shown that every solution of the game is a skew set, and subsequently that every solution is a monotonic arc connecting the faces A_M and A_N (theorem 3). It is also shown that every monotonic arc in the set of all x such that $x_i' + x_j'' \leq 1$, all $i \in M$, $j \in N$, connecting A_M and A_N is a solution. In case $g = 1$, the latter is in fact the set of all solutions. *P. Wolfe* (Santa Monica, Calif.)

BIOLOGY AND SOCIOLOGY

See also 3123, 3124.

3606:

Urbah, V. Yu. Calculation of dispersion in the statistical treatment of results of a small number of observations. Dokl. Akad. Nauk SSSR **130** (1960), 214–216 (Russian); translated as Soviet Physics. Dokl. **5**, 181–182.

3607:

Vol'kenštein, M. V. Statistical theory of reduplication of desoxyribonucleic acid. Dokl. Akad. Nauk SSSR **130** (1960), 889–892 (Russian); translated as Soviet Physics. Dokl. **5**, 186–188.

3608:

Asano, Chooichiro. Some studies of the optimum choice of dosage in biological assay. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **7** (1960), 65–78.

Consider results obtained from subjects drawn at random from a given universe, and suppose that the response to a drug is $u(x, \beta)$, where x is a dose and β shows an individual sensitivity of the subject. Then the expected response is $E\{u(x, \beta)\} = U(x)$. A difference between potencies of a test drug and a standard drug is then defined as $I(x) = U_t(x) - U_s(x)$ for the same dose x. The relation between dosage and response may be represented by a cumulative function, and the authors study the case

$$p(x, m, \sigma) = (2\pi)^{-1/2}\sigma^{-1}H \int_{-\infty}^{x} \exp[-(x-m)^2/2\sigma^2]dx.$$

Topics included are: line-fitting for the dose-response relation; choice of one dose level in direct assay; choice of two dose levels; numerical tables.

R. G. Stanton (Waterloo, Ont.)

3609:

Malý, Vladimír. Ein Vergleich der Zuverlässigkeit verschiedener Methoden zur Feststellung der Anwesenheit von Mikroben. Apl. Mat. **5** (1960), 272–281. (Czech. Russian and German summaries)

Author's summary: "Die biologischen und medizinischen Messungen sind oft durch einen recht grossen Messungsfehler gekennzeichnet; ebenso wird bei der Feststellung der Anwesenheit eines gewissen Kennzeichens auf einer Reihe von Elementen in einem bestimmten Prozentenzahl der Fälle die Anweisenheit nicht erwiesen, auch in dem Fall wenn das Kennzeichen anwesend ist. Die Feststellung durch Kombination verschiedener Methoden erhöht natürlicherweise die Zuverlässigkeit unserer Kenntnisse; es ist jedoch notwendig zu entscheiden, welche der Beobachtungsmethoden die beste ist, d. h. die die kleinste Wahrscheinlichkeit des Fehlers hat, eventuell ob eine der Methoden nicht insofern unzuverlässig ist, dass ihre Anwendung mit Rücksicht auf die Zuverlässigkeit der angewendeten Methoden unoekonomisch ist.

Vom biologischen Standpunkt aus ist es ebenfalls wichtig den Prozentsatz der Fälle abschätzen zu können, bei denen für die gegebene Kombination der Methoden die Anwesenheit des untersuchten Kennzeichens durch keine von diesen erwiesen wurde; die Kenntnis dieses Prozentsatzes ermöglicht es dem Wissenschaftler einen kritischen Standpunkt zu dem Material und den Methoden einzunehmen und führt zur Vorsicht bei Schlussfolgerungen."

3610:

Castoldi, Luigi. Attorno a una teoria di Rashevsky della respirazione cellulare. Atti Accad. Ligure **14** (1957) 18–26 (1958). (English summary)

3611:

Flückiger, M. Bevölkerungsvorausberechnung in stochastischer Darstellung. Schweiz. Z. Volkswirtschaft. Statist. **93** (1957), 417–444.

A detailed account of the most elementary ideas of stochastic population-growth theory, with some actuarial applications. *D. G. Kendall* (Oxford)

3612:

Kendall, David G. Birth-and-death processes, and the theory of carcinogenesis. Biometrika **47** (1960), 13–21.

Cells are subjected to carcinogenetic action causing some to mutate into "grey cells", whose clones are benign growths, and further causing some grey cells to mutate into "black cells", whose clones are malignant growths. Assume that the number of grey cells produced between t_1 and t_2 is Poisson with mean $\int_{t_1}^{t_2} f(t)dt$; a grey cell has probabilities λdt and μdt of splitting or dying in dt, $\mu > \lambda$; a black cell has rates L and M with $M < L$, so that malignant clones are "supercritical". Two alternative assumptions about transformation of grey to black are A: νdt is the probability that any particular grey cell will blacken in dt, or B: νdt is the probability that a grey cell is converted into one grey and one black cell in dt. It is assumed that λ, μ, L, M are constant, but ν may be time-dependent. Since an experimenter may miss certain clones, there are probabilities $1-\gamma^n$ and $1-\beta^n$ that a grey or black clone is counted by the experimenter, $0 < \beta, \gamma < 1$. If we begin with a single grey cell it is shown, under A that $\phi(z, w, t) = E[z^{g_t} w^{B_t'}]$ satisfies a differential equation of the Riccati type in t, where g_t and B_t' are the number of grey cells and detected black clones at t. The generating function for the total number of black clones ever produced is

$$c(w) = (2\lambda)^{-1}\{\lambda+\mu+\nu-[(\lambda+\mu+\nu)^2-4\lambda(\mu+\nu\bar{q})]\},$$

where $\bar{q} = 1-(1-w)(1-M/L)$. For the case where grey cells are produced at random, as described above, the joint generating function for the numbers of detected grey and black clones at t is expressed in terms of the above function ϕ, and the asymptotic form of the moments is determined.

The problem and a sketch of the line of attack were suggested to the author by J. Neyman, in connection with theories of carcinogenesis [Neyman and Scott, Science **130** (1959), 303–308; MR **21** #6296]. *T. E. Harris* (Santa Monica, Calif.)

3613:

DuBois, Philip H. An analysis of Guttman's simplex. Psychometrika **25** (1960), 173–182.

Author's summary: "Applying a Spearman formula for factor loadings to a variant of the diagonal method, the Guttman simplex model is factored algebraically into $n/2$ additive factors. The finding that communalities can be discovered such that the rank of a simplex becomes $n/2$ is contradictory to Guttman's contention that the minimal rank is $n-2$. Certain matrices of 4 and 5 variables presented by Guttman as simplexes, can, in general, be considered 2-factor matrices, easily analyzed to simple structure without rotation. One example of 6 variables is factored by the method described to a 3-factor structure."

3614:

Madansky, Albert. Determinantal methods in latent class analysis. Psychometrika **25** (1960), 183–197.

Author's summary: "Some extensions of the existing determinantal methods for solving the accounting equations in latent class analysis are presented. These extensions cover more cases than previous methods, give rise to new sufficient conditions for identifiability of the latent class model, and give insight into the necessity of various sufficient conditions for identifiability. These implications to the identifiability problem are discussed."

INFORMATION AND COMMUNICATION THEORY

See also A2510, A2845, A2890, 3154.

3615:

Zakai, Moshe. A class of definitions of "duration" (or "uncertainty") and the associated uncertainty relations. Information and Control **3** (1960), 101–115.

The author suggests a class of definitions of the "bandwidth" and the "time duration" of a signal in terms of norms of L^p spaces. These definitions correspond in several special cases to definitions which are already in use or which have clear heuristic value. Some mathematical properties of the new class of definitions are derived and discussed. The author also suggests a generalization to signals which depend on more than one variable. *I. Kay* (New York)

3616:

Max, Joel. Quantizing for minimum distortion. Trans. IRE **IT-6** (1960), 7–12.

Let x be the input to a quantizer with a finite number, N, of levels; and let $p(x)$ be the probability density function of x. Such a system may be described by specifying the end points x_k of the N input ranges and the corresponding output levels y_k. The author chooses the x_k's and y_k's such that the distortion is minimized. (The distortion is defined as the expected value of some function f of the quantization error ε (= input minus output).) Numerical results are obtained for $f(\varepsilon) = \varepsilon^2$, $p(x)$ normal, and various values of N. *K. S. Miller* (New York)

3617:

Korolev, L. N. Coding and code compression. J. Assoc. Comput. Mach. **5** (1958), 328–330.

Translation of the article in Dokl. Akad. Nauk SSSR **113** (1957), 746–747 [MR **22** #653].

3618:

Kelly, John L., Jr. A class of codes for signaling on a noisy continuous channel. Trans. IRE **IT-6** (1960), 22–24.

If U is the set of all input words of a fixed length for a given channel and P is a probability distribution on U, the corresponding K-word random code is usually defined as the random K-component vector whose components are independently distributed in U according to P. The essence of this article is the observation that for a channel whose set of input and output words of length N are both represented by the unit N-cube, whose defining probability

functions have densities, and whose capacity achieving input probability distribution has a density, Shannon's proof [C. E. Shannon, Information and Control **1** (1957), 6–25; MR **19**, 1148] of the coding theorem goes through with only one slight modification, if in the above definition of random code 'independently' is replaced by 'pairwise independently'. The author exhibits a simple Monte Carlo technique generating a large class of random codes, which he calls random additive codes, satisfying this weaker definition. *A. Feinstein* (Urbana, Ill.)

3619:

Bose, R. C.; Ray-Chaudhuri, D. K. On a class of error correcting binary group codes. Information and Control **3** (1960), 68–79.

The authors introduce the binary ($p=2$, $m=1$) case of the following large and important class of error-correcting codes. Let p be a prime, m a positive integer, n a positive integer not divisible by p and e a positive integer at most $n/2$. Let α be a primitive nth root of unity in an extension of the p^m element field K. Let $f_i(x)$ be the minimum polynomial in $K[x]$ of α^i, $i=1, \cdots, 2e$, and let $g(x)= g_kx^k+\cdots+g_0=(1-x^n)/\text{L.C.M.}\ \{f_1, \cdots, f_{2e}\}$, $g_k\neq 0$. Let C denote the set of all n-tuples $c_1, \cdots, c_n$ of elements of K which satisfy $g_0c_j+g_1c_{j-1}+\cdots+g_kc_{j-k}=0$ for $j=k+1, \cdots, n$. Then C is an e-error correcting code in the sense that any two of its members differ in more than $2e$ components. W. W. Peterson [Trans. IRE **IT-6** (1960), 459–470] has discovered an efficient decoding procedure for the binary codes, and D. Gorenstein and the reviewer in an article submitted to the SIAM Journal have constructed a somewhat analogous decoding procedure for the general class. *N. Zierler* (Pasadena, Calif.)

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See also A2767.

3620:

Ruubel', H. V. Criterion of control inaccuracy. Avtomat. i Telemeh. **20** (1959), 856–859. (Russian. English summary)

The author expands a criterion function into a power series and thus obtains a series of approximate criteria for the performance of a system.
R. Bellman (Santa Monica, Calif.)

3621:

Fuller, A. T. Optimization of non-linear control systems with transient inputs. J. Electronics Control (1) **8** (1960), 465–479.

Author's summary: "The nature of the optimum controller is investigated for control systems which are subject to saturation. It is shown that for a wide class of performance criteria the optimum controller simply generates an instantaneous non-linear function of the input phase coordinates and of the output phase coordinates. For relay control systems, the optimum controller is characterized by a switching surface in the phase space. For special cases, the dimensions of the phase space can be reduced in number by using error coordinates. These results systematize and generalize several known results, and explain the starting points of some recent abstract papers."

3622:

Sawaragi, Yoshikazu; Sunahara, Yoshifumi. The statistical studies on the response of automatic control systems with a non-linear element of zero-memory type. IV. On the statistical evaluation of the closed loop response of non-linear control systems subjected to a random command signal contaminated by a random noise. Tech. Rep. Engrg. Res. Inst. Kyoto Univ. **9** (1959), 209–222.

In an earlier paper [same Rep. **8** (1958), 195–218; MR **21** #4067] the authors have discussed statistical properties of the signals in a control system which contains a nonlinear element of the zero-memory type, and which is subject to a sinusoidal command signal and to a Gaussian random noise. The present paper gives a similar discussion for the case in which both the command signal and the noise are Gaussian random functions. Particular attention is given to the statistical effects due to (a) the ratio of the central frequencies of the command signal and of the noise, (b) the values of the system parameters, and (c) the signal-to-noise ratio of the input to the system.
L. A. MacColl (New York)

3623:

Sawaragi, Yoshikazu; Sunahara, Yoshifumi. The statistical studies on the response of automatic control systems with a non-linear element of zero-memory type. V. Some results on the synthetical studies of control systems with random inputs. Tech. Rep. Engrg. Res. Inst. Kyoto Univ. **10** (1960), 1–17.

In the first part of this paper the theory of control systems of the title that the authors have developed in previous papers of this series [same Rep. **8** (1958), 95–126; 195–218; **9** (1959), 77–96; MR **20** #793; **21** #4067, 4068] is applied to a particular system, and the effects of the values of various parameters on the statistical properties of the signals is studied in detail. In the second and more important part of the paper it is suggested that the performance of a control system may be improved, from the statistical point of view, by the incorporation in the system of a suitably chosen nonlinear element. A procedure for determining the optimum nonlinear element in a class of elements having analytic characteristics is described, and two illustrative examples are worked out.
L. A. MacColl (New York)

3624:

Sawaragi, Yoshikazu; Sunahara, Yoshifumi. Modification of the equivalent gains of non-linear elements considering the probability density function of the response of non-linear control systems subjected to a Gaussian random input. Tech. Rep. Engrg. Res. Inst. Kyoto Univ. **10** (1960), 43–58.

Having in view applications to the theory of nonlinear control systems, the authors have computed in an earlier paper [same Rep. 8 (1958), 95–126; MR **20** #793] equivalent gains for a nonlinear element, of the zero-memory type, with inputs containing random components. The computations were based on the assumption that all of the random signals involved are approximately Gaussian. In the present paper some control systems, containing

nonlinear elements and with Gaussian random inputs, are considered, which are such that the equivalent gains can be computed without the aid of the assumption that the signals are approximately Gaussian throughout the system. A comparison of the equivalent gains with those computed earlier shows that the corrections resulting from the more accurate computations are appreciable, but mostly rather small. The results are said to be confirmed by certain experiments which are described briefly.

L. A. MacColl (New York)

3625:

Rozenvasser, E. N. Concerning stability problem of nonlinear control systems. Avtomat. i Telemeh. **20** (1959), 702–707. (Russian. English summary)

The author extends some results of Lure concerning the boundedness of solutions of systems of the form $x_i'=\sum_{j=1}^N a_{ij}x_j+h_i f(s)$, where s is a linear function of the x_i. Lyapunov's second method is used.

R. Bellman (Santa Monica, Calif.)

3626:

Bedel'baev, A. K. On some simplified criteria of nonlinear control systems stability. Avtomat. i Telemeh. **20** (1959), 689–701. (Russian. English summary)

3627:

Prouza, Ludvík. Zur linearen Theorie automatischer Einstellvorrichtungen. Apl. Mat. **5** (1960), 196–201. (Russian. Czech and German summaries)

Author's summary: "Im vorliegenden Artikel wird eine Methode zur Berechnung optimaler linearer automatischer Einstellvorrichtungen beschrieben und es werden einige Eigenschaften solcher Vorrichtungen angegeben. Linearisierte Einstellvorrichtungen, die früher beschrieben wurden und bei denen das Messresultat von einem einzigen Arbeitsstück benutzt wird, sind ein Spezialfall der ersteren. Es wird kurz die technische Realisierung des optimalen Filters in Rückkopplungsanordnung mittels eines Digitalrechenschemas beschrieben."

3628:

Doležal, Václav. Über einige Kriterien der Monotonie von Einschwingvorgängen in Linearsystemen. Apl. Mat. **5** (1960), 45–62. (Czech and Russian summaries)

Für Laplace-Transformierte werden Bedingungen abgeleitet, die hinreichend für einen monotonen Verlauf der zugehörigen Originalfunktionen sind. Dabei wird vorausgesetzt, daß sich das betrachtete System durch gewöhnliche Differentialgleichungen beschreiben läßt und daß die Anfangsbedingungen durch eine normierte sprunghafte Störung charakterisiert sind. Es werden weiterhin Systeme untersucht, die bei einer beliebigen monoton verlaufenden Anfangsstörung monotone Reaktionen aufweisen. Durch Einführung "vollmonotoner Funktionen" gelingt es, sowohl Kriterien für Monotonie als auch für nicht negatives Verhalten der Originalfunktionen anzugeben. Die Anwendung dieser Kriterien wird an verschiedenen Beispielen demonstriert.

K. Magnus (Stuttgart)

3629:

Sawaragi, Yoshikazu; Fukawa, Hiroshi. Statistical synthesis of finite settling time systems. Tech. Rep. Engrg. Res. Inst. Kyoto Univ. **10** (1960), 59–74.

Authors' summary: "This paper presents a statistical design method of finite settling time systems. A general procedure is described of determining, by the method of minimizing the mean square error, the pulse transfer function of a controller of a sampled data control system with a stationary random noise which has been designed to respond to a step input, a ramp input, and a higher order input without steady state error."

3630:

Kričevskiĭ, R. E. On the complexity of the realization of functions by superpositions. Dokl. Akad. Nauk SSSR **126** (1959), 1195–1198. (Russian)

Earlier estimates [J. Riordan and C. E. Shannon, J. Math and Phys. **21** (1942), 83–93; MR **4**, 151; C. E. Shannon, Bell System Tech. J. **28** (1949), 59–98; MR **10**, 617; O. B. Lupanov. same Dokl. **119** (1949), 23–26; MR **20** #7598] of complexity have been concerned with realizations of Boolean functions as networks of switching elements. A slightly different, and in some respects more general, class of realizations of functions is treated in the present paper. An index of complexity is defined for this class in a way that permits the development of a lower bound for the complexity of realizations. Although the article appears to be a rather hastily abridged version of a more complete paper, several specializations of the principal result are noted.

A superposition is defined by taking two sets of symbols $x=\{x_\mu\}$ and $L=\{L_K\}$, and associating a natural number S_K with each symbol L_K, in the latter set. The expression $L_K(\ ,\cdots,\ )$, with S_K empty places between the parentheses, is called an S_K placed elementary object, and the class of such elementary objects is denoted by σ_0.

Each symbol in $x=\{x_\mu\}$ is called a superposition of rank 0, and $L_K(A_1, A_2, \cdots, A_{S_K})$ is called a superposition of rank n where $A_1, A_2, \cdots, A_{S_K}$ are all superpositions whose maximum rank is $n-1$. The set of superpositions is denoted by σ.

An index of simplicity (complexity) I is a non-negative functional defined over the union of σ_0, σ, and x which is equal to zero on x. It is also assumed to follow the rule:

$$I[L_K(A_1, \cdots, A_{S_K})] \ge I[L_K(\ ,\cdots,\ )]+ I[A_1]+\cdots+I[A_{S_K}].$$

This rule is characteristic of realizations in terms of superpositions as treated here and in Riorden and Shannon [op. cit.], and may be contrasted with another type of realization studied by Lupanov [op. cit.] and Shannon [op. cit.] in which repeated appearances of a given expression A_i are not thought of as introducing additional complexity. It is pointed out that using the latter rule, significantly lower complexity bounds can be obtained than those which are given here.

A realization of a set $\{f\}$ of objects f by σ is defined as a mapping of σ onto $\{f\}$, and superpositions of σ which map to f are called realizations of f. Let $L(f)$ be the lower bound of indices of superpositions from σ realizing f. Then if D_h is an arbitrary finite subset of $\{f\}$ the expression

$L(D_h)$ will be taken to mean the maximum $L(f)$ for f in D_h.

The principal result in the paper is an expression for a lower bound on $L(D_h)$. This bound is valid only if the number τ of times that any single placed elementary object may be used iteratively in the realization of $\{f\}$ is taken as being finite. Let K_l = the number of l-placed elementary objects and be finite for all l, p_l = the minimum index of an l-placed elementary object,

$$\rho(n) = \inf_{p_l \leq n, l \geq 2} p_l/(l-1),$$
$$\psi(n) = \sup_{l \geq 2, p_l \leq n} \log_2 K_l/p_l \quad \text{if} \quad p_1 = 0,$$
$$= \sup_{l \geq 1, p_l \leq n} \log_2 K_l/p_l \quad \text{if} \quad p_1 > 0,$$

$m(h)$ = the magnitude of the set D_h, $j(h)$ = the number of different symbols from $\{x_\mu\}$ occurring in superpositions realizing D_h. Assume that $p_1 \geqq 0$, $p_l > 0$, $l = 2, 3, \cdots$ and that $\lim_{l\to\infty} p_l = \infty$ if σ_0 is infinite. If there is a sequence of classes D_h such that $j(h)$ is finite for any h and as $h \to \infty$, $m(h) \to \infty$, and $\log m(h)/\log j(h) \to \infty$, then for every $\varepsilon > 0$ there exists an $h(\varepsilon)$ such that $L(D_h) > n_0(1-\varepsilon)$ for all $h > h(\varepsilon)$, where n_0 is any solution of the inequality

$$n(\log_2 [c \cdot j(h)]/\rho(n) + \psi(n)) \leqq \log_2 m(h), \quad c = \text{const.}$$

Also the proportion of elements of D_h realized by superpositions having index less than $n_0(1-\varepsilon)$ may be made as small as desired by taking h sufficiently large.

If D_h is taken as the set of Boolean functions depending on h arguments the above result specializes to become

$$L(D_h) > \frac{2^h}{\log_2 h}(1-\varepsilon).$$

D. E. Muller (Urbana, Ill.)

3631:

Moisil, Grigore Constantin. Teoria algebrica dei meccanismi automatici. Confer. Sem. Mat. Univ. Bari **33–34**, 31 pp. (1957).

In two lessons the author deals with the problem of finding suitable algorithms for representing and synthesizing relay networks. The paper can be considered as an introduction to further studies since every concept is explained in a very plain way and does not imply prior knowledge even of such well-known arguments as, for example, Boolean Algebra.

In the first lesson expressions are given for ordinary relays, delayed relays, latching relays, rotary switches, and manual switches. Expressions are always given for "ideal" and "real" components. The difference is that the ideal device can only be in one of its stable positions while the real one can also be in intermediate positions. The problems that the author states as fundamental in the "theory of mechanisms" are the following. (1) Mechanism analysis: Given certain inputs and initial states, to describe how the mechanism will evolve. (2) Mechanism synthesis: Given a certain program for the Mechanism describe the devices which will build it. (3) Mechanism output: It is an extention of problem (2). It applies to mechanisms which have not only inputs and internal states, but outputs as well. These are, of course, the only useful ones.

The second lesson introduces algebraic logic into the study of mechanisms. Ideal mechanisms can be fully understood with Boole's algebra. Real Mechanisms require a three-valued algebra of the kind that was introduced by Łukasiewicz in 1920. This algebra accepts the values of "true", "false", and "doubtful". They are represented by "1", "0", and "$\frac{1}{2}$". To every proposition p the following can also exist: Np (negation of p); μp (possibility of p); νp (necessity of p); ηp (impossibility of p); γp (contingency of p, non-necessity of p); and corresponding truth tables can be constructed. These tables allow the transformation of the structure formulas of multicontact circuits and possibility of checking in this way the equivalence of different multicontact circuits. The important problem of minimizing a contact network is left therefore without a constructive method but a verification method is suggested.

The bibliography concerns mainly Rumanian and Russian works. Many American and European authors are named. *G. Sacerdoti* (Borgolombardo)

3632:

Moisil, Gr. C. ★Scheme cu comandă directă cu contacte şi relee. [Schemes with direct command with contacts and relays.] Comisia de Automatizări, Institutul de Matematici, Monografii asupra Teoriei Algebrice a Mecanismelor Automate. Editura Academiei Republicii Populare Romîne, 1959. 205 pp. (mimeographed) Lei 9.65.

Nombre de travaux élaborés dans ces dernières années par l'auteur et ses élèves [référés dans MR **19**, 375 et 376; Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1 (49)** (1957), 87–97; MR **20** #5709], à la suite de la parution du livre *Théorie des schémas à contacts et relais* par M. A. Gavrilov [*Teoriya releĭno-kontaktnyh shem*, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1950; MR **12**, 225], constituent le sujet de ce manuel. Le livre est sousdivisé en X chapitres contenant l'exposition théorique tandis que le chap. XI est consacré à la présentation historique du problème et à la bibliographie. On y remarque aussi la présence de nombreux exercices (p. 105 et suiv.) et d'exemples de schémas suivis par des solutions détaillées (p. 157 et suiv.). *D. Mangeron* (Iaşi)

3633:

Lee, C. Y. Automata and finite automata. Bell System Tech. J. **39** (1960), 1267–1295.

The author uses the term W-machine to denote a machine modified from H. Wang [J. Assoc. Comput. Mach. **4** (1957), 61–92; MR **20** #4492], which differs from a Turing machine in having instructions (erase, write, conditional transfer, and left and right motion by one tape square), and whose internal description is given as a program rather than a table of states and transitions. He gives explicit procedures for transforming in either directions between W-machine programs and the customary descriptions of Turing machines, preserving computational equivalence between the two kinds of machines. A universal Turing machine having two symbols and only 76 states is constructed by the use of W-machines. A short proof of Kleene's theorem characterizing finite automata is given, and some new results about incompletely specified finite automata are obtained, both in

terms of W-machines. The use of proofs expressed in terms of W-machines apparently reduces indirectness and obscurity in several instances.

E. F. Moore (Murray Hill, N.J.)

3634:

Hartmanis, J. Symbolic analysis of a decomposition of information processing machines. Information and Control **3** (1960), 154–178.

Given a finite-state synchronous sequential machine, the author determines (in terms of set-theoretic results on partitions) necessary and sufficient conditions that it be decomposable into two simpler machines, such that each have the same inputs, and such that the corresponding outputs of the simpler machines, when fed into "and" circuits, will produce the outputs of the original machine. This algebraic characterization of the structure of such machines could be used to break down the problem of the circuit synthesis of the original machine into two smaller circuit synthesis problems for simpler machines. When a decomposition of this form exists, this paper leads to a method of obtaining explicitly the factor machines.

E. F. Moore (Murray Hill, N.J.)

conditional necessary and sufficient conditions that it be decomposable into two simpler machines, such that each have the same inputs and such that the corresponding outputs of the simpler machines, when fed into "and" circuits, will produce the outputs of the original machine. This algebraic characterization of the structure of state machines could be used to break down the problem of the overall synthesis of the original machine into two smaller circuit synthesis problems for simpler machines. When a decomposition of this form exists, this paper leads to a method of obtaining explicitly the latter machines.

[illegible] (Murray Hill, N.J.)

[illegible]

terms of B machines. The use of matrix expressed in terms of B machines apparently relieves tediousness and obscurity in several instances.

[illegible] (Murray Hill, N.J.)

Hartmanis, J. Symbolic analysis of a decomposition of information processing machines. Information and Control 3 (1960), [illegible].

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